Bachelor of Science
(B.Sc. - PCM)

MATHEMATICAL PHYSICS -I
(DBSPSE201T24)

Self-Learning Material
(SEM -II)

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COURSE INTRODUCTION

Mathematical physics serves as the language of nature, allowing us to formulate and solve complex physical problems using mathematical frameworks. From classical mechanics to quantum theory, from fluid dynamics to electromagnetism, the marriage of mathematics with physics has propelled our understanding of the universe forward in profound ways.

The course is divided into 10 units. Each Unit is divided into sub topics.

The Units provide students with a comprehensive understanding of the Fourier Series is a powerful mathematical tool which is used to represent periodic functions as sums of simpler sine and cosine functions. This concept is fundamental in the fields of signal processing, vibration analysis, and heat transfer, among others.

There are sections and sub-sections inside each unit. Each unit starts with a statement of objectives that outlines the goals we hope you will accomplish. Every segment of the unit has many tasks that you need to complete. We wish you pleasure in the Course.

Course Outcomes: After completion of the course, the students will be able to

1. Recall the concepts of concept of Fourier series and how it represents periodic functions as sums of sine and cosine functions.
2. Explain the Fourier series expansion of a given periodic function, including both the trigonometric and complex forms.
3. Apply Fourier series to analyze and process signals in engineering, such as filtering and signal reconstruction.
4. Analyze and understand what Bessel functions are, including the different types, such as Bessel functions of the first kind $J_n(x)$, Bessel functions of the second kind $Y_n(x)$, and their respective properties.
5. Evaluate asymptotic forms of Bessel functions for large and small arguments.
6. Create methods to the relationship of Bessel functions to other special functions and solutions, such as spherical Bessel functions and Hankel functions.

Acknowledgements:

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Unit - 1

Introduction to Fourier Series

Learning Objective

- Define and understand the concept of Fourier series: Grasp the basic idea of representing periodic functions as sums of sine and cosine terms.
- Recognize the importance of Fourier series: Understand the significance and applications of Fourier series in various fields such as signal processing, heat transfer, and vibration analysis.
- Understand periodic functions: Learn about periodic functions and their properties.
- Learn trigonometric identities and integrals: Become proficient in using trigonometric identities and integrals necessary for deriving Fourier coefficients.

Structure

1.1 Fourier Series

Fourier Series is a powerful mathematical tool which is used to represent periodic functions as sums of simpler sine and cosine functions. This concept is fundamental in the fields of signal processing, vibration analysis, and heat transfer, among others. Here's a basic introduction to get you started:

A Fourier Series expresses a periodic function f(x) as a sum of sines and cosines:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

where:

- a_0 ,an, and bn are Fourier coefficients.

- The function f(x) is periodic with period T.

Why Use Fourier Series?

1. **Simplification**: Break down complex periodic functions into simple components.
2. **Analysis**: Analyze the frequency components of signals.
3. **Signal Processing**: Essential in digital signal processing, image compression, and telecommunications.

Fourier Analysis for Periodic Functions

Laurent expansions are used to represent analytic functions in Fourier series. Further basic findings in the harmonic analysis and the representation of rapidly decreasing functions by Fourier integrals and C∞ periodic functions by Fourier series, are obtained through the application of elementary complex analysis. These are transcendentally beautiful, ancient thoughts.

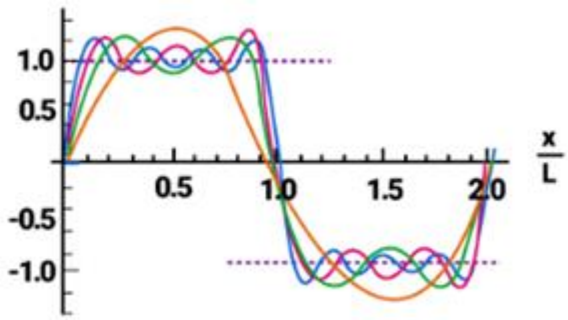

Figure1.1 Square Wave

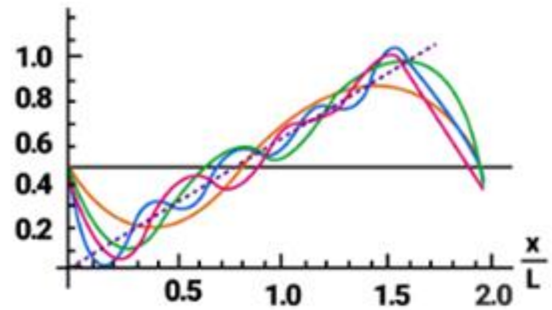

Figure1.2 Sawtooth Wave

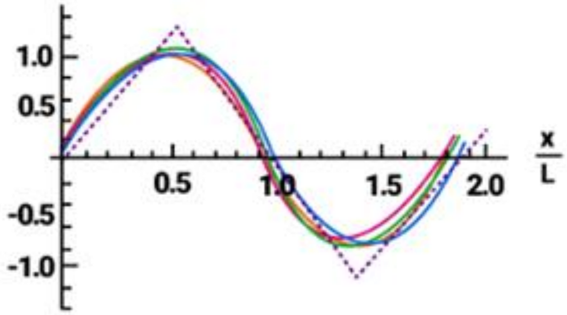

Figure1.3 Traingle Wave

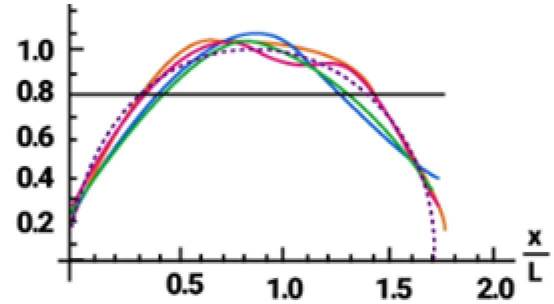

Figure1.4 Semicircle Wave

If for every x in the domain of f, f(x+L) = f(x), then the function is periodic with period L. The term "fundamental period" refers to the lowest positive value of L.

While sin x and cos x are trigonometric functions, which have basic periods of 2π, are instances of periodic functions, tan x is periodic, having a fundamental period of F. "Continuous function" refers to a periodic function with arbitrary period L.

Any linear combination of the functions $f_1,..., f_n$ that are periodic of period L may be easily verified.

$$c_1 f_1(x) + \ldots + c_n f_n(x)$$

is also periodic. Furthermore, if the infinite series

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \, cos \frac{n\pi x}{L} + b_n \, sin \frac{n\pi x}{L}$$

A function that converges to an existing 2L-periodic function will be periodic with period 2L if it does so for every x. Fourier series analysis will benefit from an understanding of two symmetry features of functions.

1.2 Even and Odd Function

When dealing with Fourier series, understanding whether a function is even or odd can simplify the calculation of Fourier coefficients. Let's review the definitions of even and odd functions, and then see how they affect the Fourier series.

- **Even Function:** A function f(t) is even if f(−t)=f(t). The graph of an even function is symmetric with respect to the y-axis.
- **Odd Function:** A function f(t) is odd if f(−t)=−f(t). The graph of an odd function is symmetric with respect to the origin.

Fourier Series for Even and Odd Functions

Even Functions

For an even function f(t):

- The sine terms bn in the Fourier series will all be zero because sin(2πnt\T) is an odd function, and the product of an even function and an odd function over a symmetric interval is zero.

The Fourier series for an even function is thus:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right)$$

The coefficients are calculated as:

1. a_0:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

2. a_n:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right) dt$$

Odd Functions

For an odd function f(t):

- The cosine terms an in the Fourier series will all be zero because cos(2πnt\T) is an even function, and the product of an odd function and an even function over a symmetric interval is zero.

The Fourier series for an odd function is thus:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

The coefficients are calculated as:

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right) dt$$

Odd functions exhibit symmetry about the origin, while even functions exhibit symmetry around the y-axis.

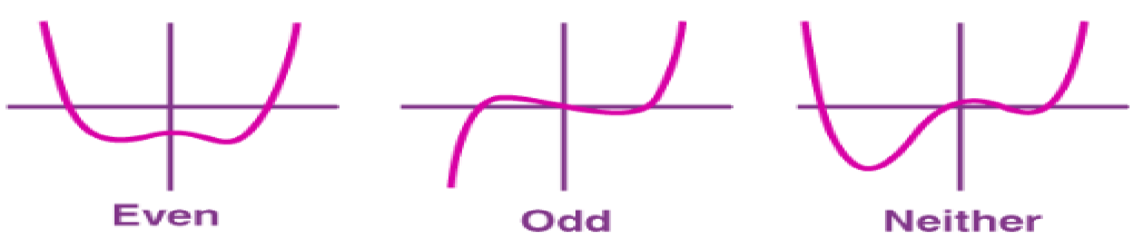

Figure 1.5 : Graphical Representation of Fourier Series

Example 1: Even Function

Consider f(x) = cos(x) which is an even function with period 2π.

The Fourier Series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

- $a_0 = \frac{1}{2\pi} \int_0^{2\pi} \cos(x)\, dx = 0$
- $a_n = \frac{2}{2\pi} \int_0^{2\pi} \cos(x) \cos(nx)\, dx$

for n=1

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} \cos(x) \cos(x)\, dx = 1$$

for n ≠ 1

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos(x) \cos(nx)\, dx = 0$$

Thus, the Fourier Series is cos(x)

Example 2: Odd Function

Consider f(x) = sin(x) which is an odd function with period 2π

The Fourier Series is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

For Sin(x)

- $b_n = \frac{2}{2\pi} \int_0^{2\pi} \sin(x) \sin(nx)\, dx$

For $n = 1$:

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} \sin(x) \sin(x)\, dx = 1$$

For $n \neq 1$:

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \sin(x) \sin(nx)\, dx = 0$$

1.3 Summary

- Fourier series apply to functions f(x) that repeat themselves over a fixed interval [a,a+L].
- These functions can be decomposed into a series of sinusoidal (or complex exponential) components.
- Fourier transform expresses a function in terms of its frequency components rather than its periodicity.

1.4 Keywords

- Fourier series
- Odd Function
- Even Function
- Periodic Functions

1.5 Self Assessment

1. What are the conditions under which a function can be represented by a Fourier series?
2. What are the Fourier coefficients a_0, a_n, and b_n?
3. What is the significance of orthogonality in Fourier series?
4. How does the period L of a function affect its Fourier series?
5. Describe the process of computing Fourier coefficients a_n and b_n.
6. What is Parseval's theorem?
7. How can Fourier series be extended to represent non-periodic functions?
8. Discuss the practical applications of Fourier series in engineering and physics.
9. What are some numerical methods or software tools used to compute Fourier series?

1.6 Case Study

Consider a thin rod of length L with insulated ends. The temperature distribution along the rod is governed by the heat equation: $u_t=\alpha.u_{xx}$ where u(x,t) represents the temperature at position x and time t, and α is the thermal diffusivity constant.

- Solve the spatial part X(x) of the separated solution using the derived PDEs and boundary conditions.

- Solve the temporal part T(t) of the separated solution using the initial condition u(x,0)=f(x).

1.7 References

- Rukmangadachari, E. (2011). Engineering Mathematics - II:. India: Pearson Education India.
- Kishan, H. (2006). Differential Equations. India: Atlantic Publishers & Distributors.

Unit - 2

Convergence and Properties of Fourier Series

Learning Objective

- Learn to calculate the coefficients a_0, a_n, and b_n for a given periodic function using integrals.
- Grasp how the properties of even and odd functions simplify the computation of Fourier coefficients.
- Formulate the Fourier series of a given periodic function using the computed coefficients.
- Understand the conditions under which a Fourier series converges to the original function.

Structure

2.1 Introduction

A few words that will come up in the remainder of the section need to be defined before we can go on to the issue of convergence. When f(x)≠f(a+) and the left and right limits of the function both exist, we first state that f(x) has a jump discontinuity at x=a. Next, if the function can be divided into discrete parts and the function and its derivative, $f'(x)$, are continuous on each component, we say that f(x) is piecewise smooth. While there may be some discontinuities in a piecewise smooth function, only a limited number of jump discontinuities are permitted.

Let's consider the function,

$$f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$$

Here is a sketch of this function on the interval –L≤x≤L.

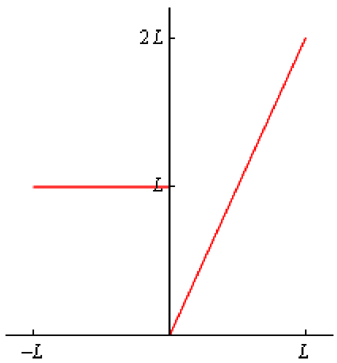

Figure 2.1 : Graphical Representation of f(x)

It should be noted that both the function and its derivative are continuous over the intervals −L≤x≤0 and 0≤x≤L. This function has a jump discontinuity at x=0 since f(0−)=L≠0=f(0+). That makes this a piecewise smooth function example. As you can see, the function is still piecewise smooth even if the function itself is not continuous at x=0 since it is a jump discontinuity.

Periodic extension is the final concept we must define.

The periodic extension of a function f(x) defined on a certain interval (hence −L≤x≤L only) is the new function that is created by repeating the function's graph on the interval to the left and right of the original function's graph on the interval.

Perhaps the simplest way to assist clarify the phrases above is to show an example of a periodic extension at this point. The function's period extension is seen in this illustration.

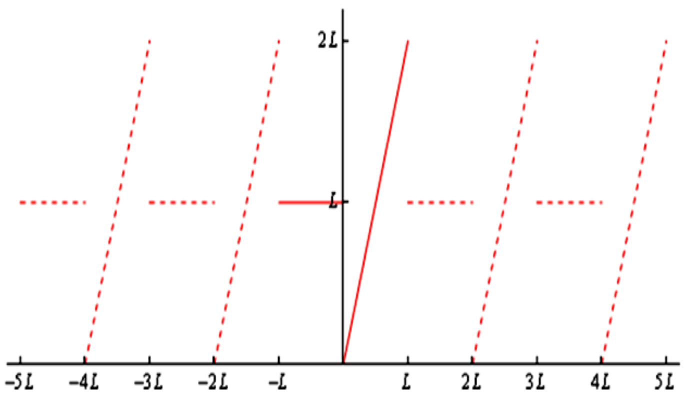

Figure 2.2 : Graphical Representation of f(x) defined on –L ≤ x ≤L

In the region –L ≤x ≤L, the solid line represents the original function. Subsequently, we obtained the periodic extension of this by grabbing this segment and replicating it at each interval of length 2L to the left and right of the initial graph. The original graph's two sets of dashed lines on either side are used to illustrate this.

It should be noted that, upon establishing the periodic extension, we have a new periodic function that is equivalent to the initial function on –L ≤ x ≤L.

Convergence of Fourier series

Let's say that f(x) is a piecewise smooth on the –L ≤ x ≤L interval. Then, f(x)'s Fourier series will converge to,

1. If the periodic extension is continuous, the periodic extension of f(x).
2. In the event that the periodic extension exhibits a jump discontinuity at x=a=a, the average of the two one-sided limits, 1/2[(a−)+f(a+)],

Let's again consider a function . In that section we found that the Fourier series of

$$f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$$

on $-L \le x \le L$ to be,

$$f(x) = L + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty} \frac{L}{n\pi}\left(1 + (-1)^n\right)\sin\left(\frac{n\pi x}{L}\right)$$

As the function exhibits a jump discontinuity at x=0, the periodic extension will likewise exhibit a jump discontinuity at this point. Accordingly, the Fourier series will converge to, at x=0.

$$\frac{1}{2}\left[f\left(0^-\right) + f\left(0^+\right)\right] = \frac{1}{2}[L + 0] = \frac{L}{2}$$

So, at x=−L

$$\frac{1}{2}\left[f\left(-L^-\right) + f\left(-L^+\right)\right] = \frac{1}{2}[2L + L] = \frac{3L}{2}$$

and at x=L

$$\frac{1}{2}\left[f\left(L^-\right) + f\left(L^+\right)\right] = \frac{1}{2}[2L + L] = \frac{3L}{2}$$

The convergence of Fourier sine/cosine series may now be quickly discussed once we have covered the convergence of a Fourier series. It was said in the previous section that the Fourier sine series for an odd function on $-L \leq x \leq L$ and the Fourier cosine series for an even function on $-L \leq x \leq L$ are merely instances of Fourier series. Based on this, we can conclude that both of these series will have the same convergence as a Fourier series.

When we examine the Fourier sine series of any function, g(x), in the interval $0 \leq x \leq L$, we may deduce that it represents the Fourier series of the odd extension of g(x), limited to that interval. Since the Fourier series is continuous on $-L \leq x \leq L$, we know that it will converge to the odd extension there as well as the average of the limits when the odd extension has a jump discontinuity. On the other hand, we know that g(x) and the odd extension are equivalent on $0 \leq x \leq L$.

Example 1: Determine the Fourier series expansion of $f(x) = e^x$, within the limit $[-\pi, \pi]$.

Solution:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x dx$$
$$= \frac{e^{\pi} - e^{-\pi}}{2\pi} \ .$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x cos(nx) dx$$
$$= \frac{1}{\pi} \frac{e^x}{1+n^2} [\cos(nx) + n\sin(nx)]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi(1+n^2)} [e^{\pi}(-1)^n) - e^{-\pi}(-1)^n)] \ .$$
$$b_n = \frac{1}{\pi} \int e^x sin(nx) dx$$
$$= \frac{e^x}{\pi(1+n^2)} [\sin(nx) - n\cos(nx)]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi(1+n^2)} [e^{\pi}(-n(-1)^n) - e^{-\pi}(-n)(-1)^n] \ .$$
$$\frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)} [cosnx - nsinnx]$$

Example 2: Determine the Fourier series of f(x) = 1 for limits [– π, π] .

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \cdot cosnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \cdot sinnxdx.$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos\ nx + \sum_{n=1}^{\infty} b_n sin\ nx$$

$$f(x) = \pi + 0 + 0$$

$$f(x) = \pi$$

Example 3: Consider f(x) = x^2 for the limits [– π, π]. Find its Fourier series expansion.

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot cosnxdx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot sinnxdx.$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos\ nx + \sum_{n=1}^{\infty} b_n sin\ nx$$

$$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} a_n cos\ nx + 0.$$

$$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \frac{4\pi cosn\pi cosnx}{n2}.$$

Example 4: Find Fourier series expansion of f(x) = 4-3x for the limits [– 1, 1].

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 4 - 3x\, dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 4 - 3x \cdot cosnxdx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 4 - 3x \cdot sinnxdx.$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos\ nx + \sum_{n=1}^{\infty} b_n sin\ nx.$$

$$f(x) = \frac{1}{2\cdot 1} \cdot 8 + \sum_{n=1}^{\infty} \frac{1}{1} \cdot 0 \cdot \cos\left(\frac{n\pi x}{1}\right) + \sum_{n=1}^{\infty} \frac{1}{1}\left(\frac{6(-1)^n}{\pi n}\right)\sin\left(\frac{n\pi x}{1}\right).$$

$$f(x) = 4 + \sum_{n=1}^{\infty} \frac{6(-1)^n \sin(\pi n x)}{\pi n}.$$

2.2 Properties of Fourier Series

Fourier series have several important properties that make them a powerful tool for representing and analyzing periodic functions. Here are some key properties:

1. Linearity

The Fourier series is a linear operator. If f(x) and g(x) are two periodic functions with Fourier series, and c1 and c2 are constants, then the Fourier series of c1f(x)+c2g(x) is given by the sum of the Fourier series of f(x) and g(x) scaled by c1 and c2 respectively.

If $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ **and** $g(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}$,
then $c_1 f(x) + c_2 g(x) \sim \sum_{n=-\infty}^{\infty} (c_1 c_n + c_2 d_n) e^{inx}$.

2. Periodicity

The Fourier series of a function inherits the periodicity of the original function. If f(x) is periodic with period 2π, then its Fourier series is also periodic with period 2π.

3. Symmetry Properties

Fourier coefficients have specific symmetry properties based on the symmetry of the function f(x).

- **Even Functions:** If f(x) is even, f(−x)=f(x), then its Fourier series contains only cosine terms (the sine coefficients $b_n$are zero).
- **Odd Functions:** If f(x) is odd, f(−x)=−f(x), then its Fourier series contains only sine terms (the cosine coefficients a_n are zero).

4. Orthogonality

The sine and cosine functions that make up the Fourier series are orthogonal over the interval [−π,π]. This orthogonality is the basis for deriving the Fourier coefficients.

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\delta_{mn}$$
$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi\delta_{mn}$$
$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0$$

5. Parseval's Theorem

The integral of the square of the function is proportional to the sum of the squares of the Fourier coefficients, according to Parseval's theorem. This offers a means of calculating the function's total energy (or power) using its Fourier coefficients.

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx$$

6. Convolution Property

Two periodic functions can be convolutioned and represented in terms of their Fourier series. The product of the Fourier coefficients of f(x) and g(x) determines the Fourier coefficients of the convolution (f∗g)(x), if f(x) and g(x) have Fourier coefficients of {an} and {bn}}, respectively.

Formally: $c_n = a_n . b_n$ where c_n are the Fourier coefficients of (f∗g)(x).

7. Frequency Shifting

If f(x) has a Fourier series with coefficients {an,bn}, then shifting f(x) by a phase α results in a Fourier series where the coefficients are modified by a phase factor.

If $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$, then $f(x-\alpha) \sim \sum_{n=-\infty}^{\infty} c_n e^{in(x-\alpha)}$.

8. Differentiation and Integration

The Fourier series of the derivative and integral of a function can be expressed in terms of the Fourier coefficients of the original function.

- **Differentiation:** If f(x) has Fourier coefficients $\{a_n, b_n\}$, then the Fourier coefficients of f′(x) are given by: $a_n' = nb_n$, $b_n' = -na_n$.
- **Integration:** If f(x) has Fourier coefficients $\{a_n, b_n\}$, then the Fourier coefficients of the integral of f(x) are given by: $A_n = a_n/n, B_n = b_n/n$ with an additional constant term for the indefinite integral.

2.3 Summary

- Fourier series converge under certain conditions, typically governed by the Dirichlet conditions (piecewise smoothness and bounded variation).

- Convergence behavior may vary, and pointwise convergence might not hold uniformly.
- Fourier series apply to functions f(x) that repeat themselves over a fixed interval [a,a+L].
- These functions can be decomposed into a series of sinusoidal (or complex exponential) components.

2.4 Keywords

- Fourier Series
- Convergence
- Orthogonality
- Properties of Fourier Series

2.5 Self Assessment

1. State and explain the Dirichlet conditions required for the convergence of Fourier series.
2. Differentiate between pointwise and uniform convergence of Fourier series.
3. Discuss the Gibbs phenomenon in the context of Fourier series convergence.
4. Explain the role of Fourier coefficients in determining the convergence of Fourier series.
5. State and explain Parseval's theorem for Fourier series.
6. Discuss the conditions under which Fourier series converge in L^2 norm.
7. Compare and contrast the convergence properties of Fourier series with Fourier transforms.
8. Explain the significance of uniform convergence of Fourier series in practical applications.
9. Describe strategies or methods to improve the convergence of Fourier series in numerical simulations.
10. Provide examples of engineering or scientific applications where understanding Fourier series convergence is crucial.

2.6 Case Study

Consider a smooth periodic function f(x)f(x)f(x) with Fourier series representation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

- "Investigate the rate of convergence of the Fourier series in terms of the decay of Fourier coefficients c_n."
- Compute the Fourier coefficients c_n for f(x).
- Discuss how the rate of convergence depends on the decay of $|c_n|$ as $|n| \to \infty |n|$
- Provide numerical examples or plots to illustrate the relationship between |cn| and the accuracy of the Fourier series approximation.

2.8 References

- Rukmangadachari, E. (2021). Engineering Mathematics - II:. India: Pearson Education India.
- Kishan, H. (2016). Differential Equations. India: Atlantic Publishers & Distributors.

Unit - 3

Introduction to Special Functions

Learning Objective

- To equip students with a solid understanding of the theory, properties, applications, and computational aspects of special functions, preparing them for advanced studies and practical applications in various scientific and technical fields.

Structure

3.1 Special functions

"Special functions" is a term used in mathematics to refer to a class of functions that arise in the solutions of certain types of problems in mathematical physics, engineering, and other applied sciences. These functions often have important and well-studied properties, and they frequently appear in the solutions to differential equations, integrals, and other mathematical expressions.

Power series solution of differential equation

Differential equations can be solved using the power series approach, which works on the assumption that the answer can be written as a power series. Solving linear differential equations with changing coefficients is a particularly good use for this approach.

Solution of the differential equation:

$$\frac{d^2y}{dx^2} - y = 0$$

$$y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \text{and} \quad y = e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

Ordinary Point:-

In the context of differential equations, an ordinary point is a specific type of point in the domain of a differential equation where the equation's coefficients behave nicely, meaning that the solution to the differential equation can be expressed as a power series around that point.

Consider a second-order linear differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

A point $x = x_0$ is called an **ordinary point** if the functions$a_0(x), a_1(x)$, and $a_2(x)$are analytic at $x_0$and $a_2(x) \neq 0$.

Singular point:-

In the context of differential equations, a **singular point** (or singularity) is a point at which the differential equations coefficients fail to be analytic or the leading coefficient (the coefficient of the highest-order derivative) is zero. Singular points are categorized into two main types: regular singular points and irregular singular points.

Consider the second-order linear differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

A point $x = x_0$ is a singular point if either $a_2(x) = 0$ or one of the coefficients$a_0(x), a_1(x)$, or $a_2(x)$is not analytic at x_0.

When ordinary point x=0 then solution of differential equation:

When a differential equation is said to have an "ordinary point" at x=0, it implies that the differential equation can be expressed in a form that is well-behaved and analytic at x=0x = 0x=0. Here's how we can understand and solve such a differential equation:

(i) Let the solution of given differential equation be:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_kx^k + ... = \sum_{k=0}^{\infty} a_k x^k$$

(ii) Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$,

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + ka_kx^{k-1} + ... + ... = \sum_{k=1}^{\infty} ka_kx^k$$

$$\frac{d^2y}{dx^2} = 2a_2 + 2.3\, a_3x + ... + a_k\, k(k-1)\, x^{k-2} + ... = \sum_{k=2}^{\infty} a_k \,.\, k\,(k-1)\,.\, x^{k-2}$$

(iii) Put the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\, etc.$ in the given equation.

(iv) By equating the coefficients of various power of x, find $a_0, a_1, a_2, \ldots ....$

(v) Put these values in the differential equation to get the required series solution.

Example 1: Solve $\frac{d^2y}{dx^2} + x^2y = 0.$

Solution: $$\frac{d^2y}{dx^2} + x^2y = 0 \qquad \ldots (1)$$

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^2 + a_5x^5 + a_6x^6 + a_7x^7 + a_8a^8 + \ldots + a_nx^n + \ldots \ldots (2)$

Since $x = 0$ is the ordinary point of the equation (1).

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3\,a_3\,x^2 + 4\,a_4\,x^3 + 5\,a_5\,x^4 + 6\,a_6\,x^5 + 7\,a_7\,x^6 + 8\,a_8\,x^7 + \ldots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 2\,.\,3\,a_3\,x + 3\,.\,4\,a_4\,x^2 + 4\,.\,5\,a_5\,x^3 + 5\,.\,6\,a_6\,x^4 + 6\,.\,7\,a_7\,x^5 + 7\,.\,8\,a_8\,x^6 + \ldots$$

Substituting in (1), we get

$$2\,a_2 + 2.3\,a_3\,x + 3.4\,a_4\,x^2 + 4.5\,x_5\,x^3 + 5.6\,a_6\,x^4 + 6.7\,a_7\,x^5 + 7.8\,a_8\,x^6 + \ldots$$
$$+ x^2(a_0 + a_1x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + a_5\,x^5 + a_6\,x^6 + a_7\,x^7 + \ldots) = 0$$

$$2\,a_2 + 6\,a_3x + (a_0 + 12\,a_4)\,x^2 + (a_1 + 20a_5)\,x^3 + (a_2 + 30\,a_6)\,x^4 + \ldots$$
$$+ [a_{n-2} + (n+2)(n+1)\,a_{n+2}]\,x^n + \ldots = 0$$

Equating to zero the coefficients of the various powers of x, we obtain

$$a_2 = 0, \;\; a_3 = 0$$

$$a_0 + 12a_4 = 0 \;\; i.e. \;\; a_4 = -\frac{1}{12}a_0$$

$$a_1 + 20\,a_5 = 0 \;\; i.e. \;\; a_5 = -\frac{1}{20}a_1$$

$$a_2 + 30\,a_6 = 0 \;\; i.e. \;\; a_6 = -\frac{1}{30}\,a_2 = 0 \qquad (a_2 = 0)$$

and so on. In general $\quad a_{n-2} + (n+2)(n+1)\,a_{n+2} = 0 \quad$ or $\quad a_{n+2} = -\frac{a_{n-2}}{(n+1)(n+2)}$

Putting $n = 5$, $\quad a_7 = -\frac{a_3}{6\times 7} = 0 \qquad (a_3 = 0)$

Putting $n = 6$, $a_8 = -\frac{a_4}{7\times 8} = \frac{a_0}{12\times 7\times 8}$

Putting $n = 7$, $a_9 = -\frac{a_5}{8\times 9} = \frac{a_1}{20\times 8\times 9}$

Putting $n = 8$, $a_{10} = -\frac{a_6}{9\times 10} = 0 \quad (a_6 = 0)$

Putting $n = 9$, $a_{11} = -\frac{a_7}{11\times 10} = 0 \quad (a_7 = 0)$

Putting $n = 10$, $a_{12} = -\frac{a_8}{12\times 11} = -\frac{a_0}{12\times 8\times 7\times 11\times 12}$

Substituting these values in (2), we get

$$y = a_0 + a_1 x - \frac{1}{12}a_0 x^4 - \frac{a_1}{20}x^5 + \frac{a_0}{12\times 7\times 8}x^8 + \frac{a_1}{20\times 8\times 9}x^9 - \frac{a_0}{12\times 8\times 7\times 11\times 12}x^{12} + \dots$$

$$y = a_0\left(1 - \frac{1}{12}x^4 + \frac{x^8}{12\times 7\times 8} - \frac{x^{12}}{12\times 8\times 7\times 11\times 12} + \dots\right) + a_1\left(x - \frac{x^5}{20} + \frac{x^9}{20\times 8\times 9}\dots\right) \text{ Ans.}$$

Example 2: Find the power series solution of

$$(1 - x^2)\, y'' - 2xy' + 2y = 0 \text{ about } x = 0.$$

Solution: Let,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots 1,$$

Be the required solution. Since the given equation's ordinary point is x=0, this may be expressed as

$$y = \sum_{k=0}^{\infty} a_k x^k$$

Then $\frac{dy}{dx} = \sum_{k=1}^{\infty} a_k \cdot k x^{k-1}$, $\quad \frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} a_k \cdot k(k-1)x^{k-2}$

Substituting the values of $y \cdot \frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation we get

$$(1-x^2)\Sigma a_k\, k\cdot(k-1)x^{k-2} - 2x\Sigma a_k \cdot k x^{k-1} + 2\Sigma a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k \cdot k\cdot(k-1)x^{k-2} - \Sigma a_k k(k-1)x^k - 2\,\Sigma a_k . k x^k + 2\Sigma a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k \cdot k\cdot(k-1)x^{k-2} - \Sigma[k(k-1) + 2k - 2]a_k x^k = 0$$

$$\Rightarrow \quad \Sigma a_k \cdot k\cdot(k-1)x^{k-2} - \Sigma(k^2 + k - 2)a_k x^k = 0$$

Now equating the coefficient of x^k equal to zero, we have

$\Rightarrow \qquad (k+2)(k+1)a_{k+2} - (k^2+k-2)a_k = 0$

$\Rightarrow \qquad a_{k+2} = \dfrac{k^2+k-2}{(k+2)(k+1)}a_k = \dfrac{(k+2)(k-1)}{(k+2)(k+1)}a_k$

$\Rightarrow \qquad a_{k+2} = \dfrac{k-1}{k+1}a_k$

For $k=0$ $\qquad a_2 = -a_0,\ a_3 = 0,\ a_4 = \dfrac{a_2}{3} = -\dfrac{a_0}{3},\ a_5 = \dfrac{2}{4}a_3 = 0$

For $k=4$ $\qquad a_6 = \dfrac{3}{5}a_4 = \dfrac{3}{5}\left(-\dfrac{a_0}{3}\right) = -\dfrac{a_0}{5},\ a_7 = \dfrac{4}{5}a_5 = 0,$ etc.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \dots$$

$\Rightarrow \qquad y = a_0 + a_0x - a_0x^2 + 0 - \dfrac{a_0}{3}x^4 + 0 - \dfrac{a_0}{5}x^6 + 0 + \dots$

$\Rightarrow \qquad y = a_0\left[1 - x^2 - \dfrac{x^4}{3} - \dfrac{x^6}{5} + \dots\right] + a_1x$

Solution about singular points:

When dealing with differential equations, understanding singular points is crucial as they can affect the behavior and solutions of the equation. Singular points can broadly be classified into two types: regular and irregular.

Regular Singular Points:

A regular singular point $x=x_0$ of a differential equation is characterized by the fact that the equation can be written in a standard form where $(x-x_0)$ multiplied by some function keeps the equation analytic near $x=x_0$.

Example 3: Find regular singular points of the differential equation.

$$2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (x^2-4)y = 0$$

Solution:

$$\frac{d^2y}{dx^2}+\frac{3}{2x}\frac{dy}{dx}+\frac{x^2-4}{2x^2}y=0$$

$$P_1=\frac{3}{2x} \text{ and } P_2=\frac{x^2-4}{2x^2}$$

$$Q_1=x\cdot P_1=x\left(\frac{3}{2x}\right)=\frac{3}{2}, \qquad Q_2=x^2P_2=x^2\cdot\frac{x^2-4}{2x^2}=\frac{1}{2}(x^2-4)$$

In this case, when x=0, P1 and P2 are not analytical. Consequently, x=0 is an equation's single point.

Hence x=0 is a regular singular point

3.2 Frobenius Method:

The Frobenius method is a powerful technique used to find solutions to second-order linear differential equations with variable coefficients, especially when one or both roots of the indicial equation are equal. It's particularly useful when the equation has a regular singular point at x=0.

Steps of the Frobenius Method:

1. **Identify the Differential Equation:** Start with a second-order linear differential equation of the form:

$$x^2y''(x)+p(x)xy'(x)+q(x)y(x)=0$$

   near x=0, where p(x) and q(x) are functions that may have singularities at x=0.

2. **Assume a Power Series Solution:** Assume a solution of the form:

$$y(x)=\sum_{n=0}^{\infty}a_nx^{n+r}$$

   where r is the initial guess for the root of the indicial equation and a_n are coefficients to be determined.

3. **Derive the Indicial Equation:** Substitute the power series solution into the differential equation and equate coefficients of like powers of x. This process yields a recurrence relation for the coefficients a_n. The resulting equation is called the indicial equation, which determines the roots $r_1$and r_2.

4. **Solve the Indicial Equation: The** indicial equation is typically of the form:

$$r(r-1)+p_0r+q_0=0$$

where p_0 and q_0 are coefficients obtained from the differential equation. Solve for r_1 and r_2 to find the roots.

5. **Determine the Series Solution:** Depending on the nature of the roots of the indicial equation:

   Distinct Roots r_1 and r_2: Use two linearly independent solutions of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n^{(1)} x^n \text{ and } y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n^{(2)} x^n.$$

6. **Determine Coefficients a_n and b_n:** Substitute the series solutions back into the differential equation and solve for the coefficients a_n and b_n using the recurrence relations derived in step 3.
7. **General Solution:** Form the general solution as a linear combination of the solutions found in step 5: $y(x) = C_1y_1(x)+C_2y_2(x)$ where C_1 and C_2 are arbitrary constants determined by initial conditions.

3.3 Summary

- Special functions encompass various families, including Bessel functions, Legendre functions, hyper geometric functions, gamma and beta functions, and many others. Each family has its own defining characteristics and serves specific purposes in mathematical modeling and analysis.
- Special functions often possess distinctive properties such as recurrence relations, integral representations, asymptotic behaviors, and special values at certain points. These properties are fundamental for understanding their behavior and application.
- Special functions are pivotal in both theoretical mathematics and applied sciences, providing powerful tools for modeling physical phenomena, solving complex equations, and advancing scientific knowledge across multiple domains.

3.4 Keywords

- Special functions
- Frobenius Method
- Elementary functions
- Gamma and Beta functions

3.5 Self Assessment

1. What are special functions?
2. Give an example of a special function and its application.
3. How are special functions different from elementary functions?
4. What are the computational challenges associated with special functions?
5. Why are special functions important in physics and engineering?
6. Name a special function used in statistics and its significance.
7. How are special functions classified?

3.6 Case Study

In quantum mechanics, the wavefunctions of a harmonic oscillator are expressed using Hermite polynomials.

- Describe the role of Hermite polynomials in representing the energy eigenstates of a quantum harmonic oscillator.
- How do these polynomials relate to the quantization of energy levels in the oscillator?
- Discuss any orthogonality properties of Hermite polynomials that are crucial for quantum mechanical calculations.

3.7 References

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Unit – 4

Bessel Function

Learning Objective

- Students should learn the definition of Bessel functions $J_n(x)$ and $Y_n(x)$), which are solutions to Bessel's differential equation in various forms (ordinary, modified, spherical, etc.). Understanding their recursive relationships and series representations is fundamental.
- Mastery of Bessel functions involves understanding their asymptotic behavior, zeros, orthogonality relations, and symmetry properties. These properties are crucial for their applications in various fields of physics and engineering.
- Applications in solving partial differential equations (like the wave equation, heat equation) with cylindrical or spherical symmetry.

Structure

4.1 Bessel Function

A family of unique functions known as the Bessel functions appears frequently in applied mathematics, engineering, and mathematical physics. They bear the name Friedrich Bessel after the German mathematician who originally popularized them in the early 1800s. Bessel's differential equation, which crops up in issues with fluid dynamics, heat conduction, wave propagation, and other topics, has solutions known as Bessel functions.

The Bessel functions, $J_n(x)$ and $Y_n(x)$, are canonical solutions to Bessel's differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

For many wave propagation and static potential problems, they are essential. The first type Bessel function, $J_n(x)$, is finite at the origin for any non-negative integer n, whereas the second kind Bessel function, $Y_n(x)$, has a singularity at the origin. Acoustics, fluid dynamics, electromagnetics, and other fields all use these functions.

Applications:

1. **Wave Propagation**:
   - Bessel functions describe the radial component of wave solutions in problems with circular or cylindrical symmetry, such as electromagnetic wave propagation in cylindrical waveguides.
2. **Heat Conduction**:
   - Temperature distributions in circular plates and cylinders can be modeled using Bessel functions in heat conduction problems.
3. **Vibration Analysis**:
   - Bessel functions describe the radial displacement of vibrating circular membranes or drumheads in mechanical vibrations.
4. **Quantum Mechanics**:
   - Bessel functions appear in the solutions to Schrödinger's equation for particles in cylindrical coordinates, such as in the study of atomic and molecular physics.

Computational Aspects:

- **Numerical Methods**: Special algorithms and software libraries (e.g., MATLAB, Python libraries like SciPy) are used to compute Bessel functions efficiently.
- **Series and Recurrence Relations**: Series expansions and recurrence relations are utilized for numerical computation of Bessel functions for various orders and arguments.

4.2 Bessel Function of the First Kind ($J_n(x)$)

The Bessel function of the first kind, $J_n(x)$, can be represented by the series expansion for integer order n:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

This series converges for all x. For non-integer order n, the definition involves a more general form but follows similar principles.

4.3 Bessel Function of the Second Kind ($Y_n(x)$)

The second form of Bessel function, $Y_n(x)$, which is also referred to as the Neumann function or Weber function, is defined as follows for integer n:

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$$

For non-integer n, the definition simplifies to a combination of $J_n(x)$ and $J_{-n}(x)$.

Asymptotic Forms

The following asymptotic forms apply to the Bessel functions for big x.

First Kind $J_n(x)$:

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

Second Kind $Y_n(x)$:

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

Solution of Bessel differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

$$y = \Sigma a_r x^{m+r} \text{ or } y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$$

$$\frac{dy}{dx} = \Sigma a_r (m+r) x^{m+r-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \Sigma a_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting these values in the equation, we have

$$x^2 \Sigma a_r (m+r)(m+r-1) x^{m+r-2} + x \Sigma a_r (m+r) x^{m+r-1} + (x^2 - n^2) \Sigma a_r x^{m+r} = 0$$

$$\Sigma a_r (m+r)(m+r-1) x^{m+r} + \Sigma a_r (m+r\} x^{m+r} + \Sigma a_r x^{m+r+2} - n^2 \Sigma a_r x^{m+r} = 0$$

$$\Sigma a_r [(m+r)(m+r-1) + (m+r) - n^2] x^{m+r} + \Sigma a_r x^{m+r+2} = 0$$

$$\Sigma a_r [(m+r)^2 - n^2] x^{m+r} + \Sigma a_r x^{m+r+2} = 0.$$

$$a_0\,[(m+0)^2 - n^2] = 0.$$

$$m^2 = n^2 \;\; i.e.\;\; m = n$$

$$a_1\,[(m+1)^2 - n^2] = 0 \;\; i.e.\;\; a_1 = 0.$$

$$a_{r+2}\,[(m+r+2)^2 - n^2] + a_r = 0$$

$$a_{r+2} = -\frac{1}{(m+r+2)^2 - n^2}\cdot a_r$$

$$a_3 = a_5 = a_7 = \ldots = 0, \text{ since } a_1 = 0$$

If $r = 0$, $\quad a_2 = -\dfrac{1}{(m+2)^2 - n^2}a_0$

If $r = 2$, $\quad a_4 = -\dfrac{1}{(m+4)^2 - n^2}a_2 = \dfrac{1}{[(m+2)^2 - n^2]\,[(m+4)^2 - n^2)]}a_0$

$$y = a_0 x^m - \frac{a_0}{(m+2)^2 - n^2}x^{m+2} + \frac{a_0}{[(m+2)^2 - n^2]\,[(m+4)^2 - n^2]}x^{m+4} + \ldots$$

$$y = a_0 x^m\left[1 - \frac{1}{(m+2)^2 - n^2}x^2 + \frac{1}{[(m+2)^2 - n^2]\,[(m+4)^2 - n^2]}x^4 - \ldots\right]$$

For $m = n$

$$y = a_0 x^n\left[1 - \frac{1}{4\,(n+1)}x^2 + \frac{1}{4^2\cdot 2\,!\,(n+1)\,(n+2)}x^4 - \ldots\right]$$

For $m = -n$

$$y = a_0 x^{-n}\left[1 - \frac{1}{4\,(-n+1)}x^2 + \frac{1}{4^2\cdot 2\,!\,(-n+1)\,(-n+2)}x^4 - \ldots\right]$$

Example 1: Prove that:

$$J_{-n}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma\,(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Solution: $\quad x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - n^2)\,y = 0$

$$y = a_0 x^n \left[1 - \frac{x^2}{2 \cdot 2 (n+1)} + \frac{x^4}{2.4.2^2 (n+1)(n+2)} - \dots \right.$$

$$\left. + (-1)^r \frac{x^{2r}}{(2^r r!) \cdot 2^r (n+1)(n+2) \dots (n+r)} + \dots \right]$$

$$= a_0 x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r} \cdot r! (n+1)(n+2) \dots (n+r)}$$

$$a_0 = \frac{1}{2^n \Gamma(n+1)}$$

$$J_n(x) = \frac{1}{2^n \Gamma(n+1)} \Sigma (-1)^r \frac{x^{n+2r}}{2^{2r} \cdot r! (n+1)(n+2) \dots (n+r)}$$

$$(\Gamma n+1 = n!)$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \frac{1}{\lfloor 1 \Gamma(n+2)} \left(\frac{x}{2}\right)^2 + \frac{1}{\lfloor 2 \Gamma(n+3)} \left(\frac{x}{2}\right)^4 - \frac{1}{\lfloor 3 \Gamma(n+4)} \left(\frac{x}{2}\right)^6 + \dots \right\}$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

If n replace by – n

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

Example 2: Prove that

$$J_{-n}(x) = (-1)^n J_n(x)$$

Solution:

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r! \Gamma|(r-n+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r! \Gamma(-n+r+1)} + \sum_{r=n}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r! \Gamma(-n+r+1)}$$

$$= 0 + \sum_{r=n}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

$$r = n + k$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} \left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,\Gamma(k+1)}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,k!}$$

$$= (-1)^n J_n(x)$$

Example 3: Prove that

$$x J_n' = n J_n - x J_{n+1}$$

Solution :

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n' = \sum \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}.$$

$$x J_n' = n \sum \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + x \sum \frac{(-1)^r \cdot 2r}{2 \cdot r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n\,J_n + x\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{s\,!\,\Gamma(n+s+2)}\left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n\,J_n + x\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s\,!\,\Gamma[(n+1)+s+1]}\left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= n\,J_n - x\,J_{n+1}$$

Example 4: Prove that

$$x\,J_n' = -n\,J_n + x\,J_{n-1}$$

Solution:

$$J_n = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$J_n' = \sum_{r=0}^{\infty}\frac{(-1)^r\,(n+2r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{2}$$

$$x\,J_n' = \sum_{r=0}^{\infty}\frac{(-1)^r\,(n+2r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty}\frac{(-1)^r\,[(2n+2r)-n]}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\,(2n+2r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} - n\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\,2}{r\,!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r} - n\,J_n = x\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma[(n-1)+r+1]}\left(\frac{x}{2}\right)^{(n-1+2r)} - n\,J_n$$

$$= x\,J_{n-1} - n\,J_n$$

Example 5: Prove that

$$\frac{d}{dx}[x^n\,J_n(x)] = x^n\,J_{n-1}(x)$$

Solution:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = \sum \frac{(-1)^r x^{2n+2r}}{r!\,\Gamma(n+r+1)\cdot 2^{n+2r}}$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum \frac{(-1)^r (2n+2r)\, x^{2n+2r-1}}{r!\,\Gamma(n+r+1)\cdot 2^{n+2r}}$$

$$= x^n \sum \frac{(-1)^r (n+r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1} = x \sum \frac{(-1)^r}{r!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n \sum \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)}\left(\frac{x}{2}\right)^{\overline{n-1}+2r}$$

$$= x^n J_{n-1}(x)$$

4.4 Summary

A set of special functions known as Bessel functions appears in many branches of mathematical physics and engineering. They bear the name Friedrich Bessel in honor of the German mathematician who popularized them at the beginning of the 1800s.

Bessel functions $J_n(x)$ and $Y_n(x)$ are solutions to Bessel's differential equation, which is given by:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$

where n is a parameter that determines the order of the Bessel function.

Types: There are several types of Bessel functions:

- Bessel functions of the first kind, Jn(x): These are finite at the origin x=0 for all n.
- Bessel functions of the second kind, $Y_n(x)$: These are singular at the origin x=0 for all n.

Properties:

- Bessel functions exhibit specific asymptotic behaviors for large and small arguments x.

- $J_n(x)$ has an infinite number of positive zeros. The zeros of $J_n(x)$ are used in defining the spherical Bessel functions and are crucial in many physical applications.
- Bessel functions are orthogonal over specific intervals with respect to a weight function involving x.

4.5 Keywords

- Bessel functions
- Special Functions
- Orthogonality.

4.6 Self Assessment

1. What are Bessel functions?
2. What is the differential equation satisfied by Bessel functions Jn(x) and Yn(x)?
3. What are the two main types of Bessel functions?
4. What is the asymptotic behavior of Jn(x) and Yn(x) for large x?
5. What are the special cases of Bessel functions when n is an integer?
6. Where are Bessel functions used in physics and engineering?
7. What are the orthogonality relations of Bessel functions?

4.7 Case Study

Light waves with wavelength λ pass through a circular aperture of radius a. Discuss the role of Bessel functions in calculating the diffraction pattern observed on a screen placed far from the aperture.

Describe how Bessel functions $J_1(k\rho)$ arise in the calculation of the intensity distribution of the diffracted light, where ρ is the radial distance on the screen and $k=2\pi/\lambda$.

4.8 References

- Brunt, B. v. (2014). The Calculus of Variations. Germany: Springer New York.
- Rukmangadachari, E. (2021). Engineering Mathematics - II:. India: Pearson Education India

Unit – 5

Hermite Polynomial

Learning Objective

- Understand the definition of Hermite polynomials $H_n(x)$, their recursive definition, and their explicit formula in terms of generating functions or Rodrigue's formula.
- Learn about the orthogonality properties of Hermite polynomials with respect to the weight function e^{-x2} on the real line.
- Be able to derive Hermite polynomials using Rodrigue's formula and understand its implications.
- Study the recurrence relations satisfied by Hermite polynomials and use them to compute higher-order polynomials efficiently.
- Explore applications of Hermite polynomials in physics (particularly quantum mechanics, where they arise in the solution of the quantum harmonic oscillator), probability theory (where they appear in the context of Hermite polynomials as characteristic functions), and engineering (where they are used in signal processing and orthogonal series expansions).

Structure

5.1 Hermite Polynomials

Hermite polynomials were first established by Pierre-Simon Laplace in 1810, but in a hardly recognisable form. Pafnuty Chebyshev conducted a thorough analysis of them in 1859.

As such, they weren't novel, even though Hermite was the first to identify the multidimensional polynomials in his papers in 1865.

The set of orthogonal polynomials over the domain (-∞,∞) with weighting function $e^{(-x2)}$ is known as the Hermite polynomials $H_n(x)$, as it is shown above for n = 1, 2, 3, and 4. In the Wolfram Language, Hermite polynomials are represented as Hermite H[n, x].

The Hermite polynomial $H_n(z)$ can be defined by the contour integral

$$H_n(z) = \frac{n!}{2\pi i} \oint e^{-t^2+2tz}\, t^{-n-1}\, dt,$$

The first few Hermite polynomials are

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_5(x) = 32x^5 - 160x^3 + 120x$$
$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$
$$H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$$
$$H_8(x) = 256x^8 - 3584x^6 + 13\,440x^4 - 13\,440x^2 + 1680$$
$$H_9(x) = 512x^9 - 9216x^7 + 48\,384x^5 - 80\,640x^3 + 30\,240x$$
$$H_{10}(x) = 1024x^{10} - 23\,040x^8 + 161\,280x^6 - 403\,200x^4 + 302\,400x^2 - 30\,240.$$

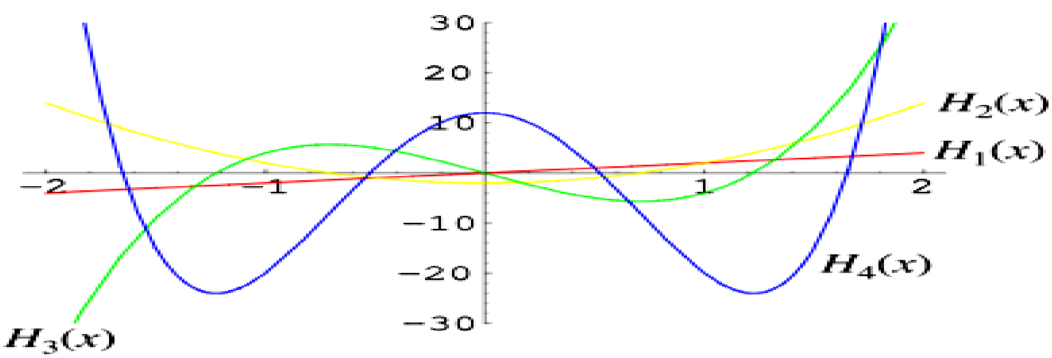

Figure 5.1: Graphical Representation of Hermite Polynomial

The Hermite functions $\psi_n(x)$ are solutions to the Hermite differential equation, and are related to the Hermite polynomials $H_n(x)$. They are given by:

$$\psi_n(x) = (\sqrt{\pi}\, 2^n\, n!)^{-\frac{1}{2}}\, e^{-\frac{x^2}{2}}\, H_n(x)$$

where $H_n(x)$ are the Hermite polynomials of degree n. These functions are important in quantum mechanics, particularly in the context of the quantum harmonic oscillator, and they form an orthonormal basis for the space of square-integrable functions.

5.2 Solution of Hermite's differential equation

No singularity exists in the finite plane for a Hermite differential equation. In order to solve the Hermite differential equation , we will employ the Power Series approach.

$$\frac{d^2 y}{d x^2} - 2 x \frac{d y}{d x} + \lambda y = 0.$$

Using the series method:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2 n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

$$(2 a_2 + \lambda a_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - 2 n a_n + \lambda a_n] x^n = 0.$$

$$a_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)} a_n$$

for $n = 0, 1, \dots.$

$$a_2 = -\frac{\lambda a_0}{2}$$

$$a_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)} a_n$$

$$y_1 = a_0 \left[1 - \frac{\lambda}{2!} x^2 - \frac{(4-\lambda)\lambda}{4!} x^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!} x^6 - \dots \right]$$

$$y_2 = a_1 \left[x + \frac{(2-\lambda)}{3!} x^3 + \frac{(6-\lambda)(2-\lambda)}{5!} x^5 + \dots \right].$$

After that, the generating function can be used to define the class of orthogonal polynomials that are Hermite polynomials.

$$g(x, t) = \exp\{-t^2 + 2tx\} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad .$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

$$H'_n(x) = 2nH_{n-1}(x)$$

The substitutions t → −t and x → −x in the generating function simply yield the parity relation $H_n(-x) = (-1)^n H_n(x)$. These recurrence relations quickly lead to the second order Hermite ODE as :

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

$$H_n(x) = (-1)^n \frac{d^n}{dt^n} \left[\exp\left\{x^2 - (t+x)^2\right\}\right]_{t=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

This can then be used to establish the orthogonality integral

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 0 \quad , \quad m \neq 0$$

According to this,The Hermite polynomials are not self-adjoint, but the functions $\varphi n(x) = e^{-x^2/2} H_n(x)$ are, and they satisfy

$$\phi''_n(x) + [2n + 1 - x^2]\, \phi_n(x) = 0$$

This is the equation of motion for a simple harmonic oscillator (SHO) in quantum mechanics, a significant use of Hermite polynomials.

The orthonormality condition for the Hermite polynomials needs to be determined. We proceed by squaring the generating function and multiplying by $\exp(-x^2)$:

$$e^{-x^2} e^{-s^2+2sx} e^{-t^2+2tx} = \sum_{m,n=0}^{\infty} e^{-x^2} H_m(x)\, H_n(x) \frac{s^m t^n}{m!\, n!}$$

This is now in a form to integrate over $(-\infty, \infty)$ and employ the orthogonality condition to collapse the double sum into a single one with m = n

$$\sum_{n=0}^{\infty} \frac{(st)^n}{(n!)^2} \int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2 \, dx \;=\; \int_{-\infty}^{\infty} e^{-x^2-s^2+2sx-t^2+2tx} \, dx$$

$$= e^{2st} \int_{-\infty}^{\infty} e^{-(x-s-t)^2} \, dx \;=\; \sqrt{\pi}\, e^{2st} \;=\; \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n (st)^n}{n!}$$

Equating the coefficients term by term yields the normalization constraint

$$\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2 \, dx \;=\; 2^n \sqrt{\pi}\, n!$$

The quantum mechanical SHO with a potential energy V = $m\omega^2$ z $^2/2$ is described by the Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi(z) + \frac{m\omega^2 z^2}{2} \Psi(z) \;=\; E\Psi(z)$$

Here ω is the angular frequency of the corresponding classical oscillator. Rescaling the spatial coordinate by x = αz with , the ODE can be written in the form (for λ = 2E/ω)

$$\frac{d^2\psi(x)}{dx^2} + (\lambda - x^2)\, \psi(x) \;=\; 0 \quad , \quad \psi(x) \;\equiv\; \Psi(z/\alpha)$$

5.3 Summary

- They are solutions to the quantum harmonic oscillator in quantum mechanics.
- They arise in the context of the Hermite series expansions and in the study of Gaussian integrals.
- They have connections to Hermite functions in higher dimensions and to special functions such as the confluent hypergeometric function.
- Quantum mechanics, statistical mechanics, and the theory of harmonic oscillators
- Hermite polynomials appear as the characteristic functions of normal distributions.
- Used in signal processing, control theory, and other fields requiring orthogonal polynomials.
- Hermite polynomials exhibit specific asymptotic behavior for large n, often approximated by Airy functions in certain contexts.

5.4 Keywords

- Hermite polynomials
- Hermite series
- Generating Functions
- Rodrigue’s formula
- Orthogonality

5.5 Self Assessment

1. How are Hermite polynomials $H_n(x)$ defined?
2. What weight function do Hermite polynomials satisfy orthogonality with respect to?
3. What is the recurrence relation satisfied by Hermite polynomials?
4. What is the generating function for Hermite polynomials?
5. In which scientific disciplines do Hermite polynomials find significant applications?
6. What are the initial conditions for Hermite polynomials $H_0(x)$ and $H_1(x)$?
7. How do Hermite polynomials behave asymptotically for large nnn?
8. How do Hermite polynomials generalize to multiple dimensions?

5.6 Case Study

1. Explain how Hermite polynomials arise in the context of solving the quantum harmonic oscillator problem. Discuss their significance in this field.
2. Hermite polynomials appear as characteristic functions of normal distributions. Explain this connection and how Hermite polynomials facilitate computations in probability theory.

5.7 References

- Brunt, B. v. (2014). The Calculus of Variations. Germany: Springer New York.
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Unit – 6

Legendre Functions

Learning Objective

- Understand the definition of Legendre polynomials $P_n(x)$ and associated Legendre functions $P_n^m(x)$.
- Learn the explicit forms of these functions and their generating functions.Comprehend the orthogonality properties of Legendre polynomials over the interval [−1,1] with respect to the weight function 1.
- Explore the orthogonality relations of associated Legendre functions on appropriate intervals with specific weight functions.
- The learning objectives of studying Legendre functions, which include Legendre polynomials and associated Legendre functions, typically involve understanding their mathematical properties, their role in solving differential equations, and their applications in physics and engineering.

Structure

6.1 Legendre's Equation

The differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is known as Legendre's equation. The above equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n\,(n+1)\,y = 0 \qquad n\ \varepsilon\ I$$

It is possible to integrate this equation in a sequence of x's ascending or descending powers. That is, series in ascending or descending powers of x that meet the equation may be identified. Let be the sequence of x in descending powers.

$$y = x^m (a_0 + a_1 x^{-1} + a_2 x^{-2} + \ldots)$$

$$y = \sum_{r=0}^{\infty} a_r x^{m-r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m-r)^{m-r-1}$$

$$\frac{d^2y}{dx^2} = \sum^{\infty} a_r (m-r)(m-r-1)\, x^{m-r-2}$$

$$(1-x^2)\sum_{r=0}^{\infty} a_r (m-r)(m-r-1)\, x^{m-r-2} - 2x\sum_{r=0}^{\infty} a_r (m-r)\, x^{m-r-1} + n\,(n+1)\sum_{0}^{\infty} a_r x^{m-r} = 0$$

or $$\sum_{r=0}^{\infty} a_r (m-r)(m-r-1)\, x^{m-r-2} + \{n\,(n+1) - 2\,(m-r) - (m-r)(m-r-1)\}\, x^{m-r} a_r = 0$$

or $$\sum_{r=0}^{\infty} [(m-r)(m-r-1)\, x^{m-r-2} + \{n\,(n+1) - (m-r)(m-r+1)\}\, x^{m-r}]\, a_r \equiv 0$$

But $a_0 \neq 0$, as it is the coefficient of the very first term in the series.

Hence $(n+1) - m\,(m+1) = 0$

$$n^2 + n - m^2 - m = 0 \quad \text{or} \quad (n^2 - m^2) + (n - m) = 0$$

$$(n-m)(n+m+1) = 0$$

which gives $m = n$ or $m = -n-1$

Next, equating to zero the coefficient of x^{m-1} by putting $r = 1$,

$$a_1 [n\,(n+1) - (m-1)\,m] = 0$$

or $$a_1 [(m+n)(m-n-1)] = 0$$

which gives $a_1 = 0$

Since $(m+n)(m-n-1) \neq 0$.

$$(m-r)(m-r-1)a_r + [n(n+1)-(m-r-2)(m-r-1)]a_{r+2} = 0$$

Now $n(n+1)-(m-r-2)(m-r-1) = n^2+n-(m-r-1-1)(m-r-1)$

$$= -[(m-r-1)^2-(m-r-1)-n^2-n]$$

$$= -[(m-r-1+n)(m-r-1-n)-(m-r-1+n)]$$

$$= -[(m-r-1+n)(m-r-1-n-1)]$$

$$= (m-r+n-1)(m-r+n-2)$$

or $\quad (m-r)(m-r-1)a_r-(m-r+n-1)(m-r-n-2)a_{r+2} = 0$

or
$$a_{r+2} = \frac{(m-r)(m-r-1)}{(m-r+n-1)(m-r-n-2)}a_r$$

$$a_1 = a_3 = a_5 = a_7 = \ldots = 0$$

For two cases there arises following three cases:

Case 1: When m=n

$$a_{r+2} = -\frac{(n-r)(n-r-1)}{(2n-r-1)(r+2)}a_r$$

$$a_2 = -\frac{n(n-1)}{(2n-1)2}a_0,$$

$$a_4 = -\frac{(n-2)(n-3)}{(2n-3)\times 4}a_2 = \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4}a_0$$

$$a_1 = a_3 = a_5 = \ldots = 0$$

$$y = a_0\left[x^n-\frac{n(n-1)}{(2n-1)\cdot 2}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4}\cdot x^{n-4}-\ldots\right]$$

Case 2: When m =-(n+1)

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)}a_0;$$

$$a_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}a_0$$

$$y = a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}\cdot x^{-n-5}+\right.$$

Legendre's Function of the Second Kind i.e. $Q_n(x)$.

Another solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0$$

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}+\dots\right]$$

$$a_0 = \frac{n!}{1.3.5\dots(2n+1)}$$

$$Q_n(x) = \frac{n!}{1.3.5.\dots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}+\dots\right]$$

6.2 Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof

Let $v = (x^2-1)^n$

$$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x)$$

$$(x^2-1)\frac{dv}{dx} = 2n(x^2-1)^n x.$$

$$(x^2-1)\frac{dv}{dx} = 2nvx$$

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}}+{}^{(n+1)}C_1(2x)\frac{d^{n+1}v}{dx^{n+1}}+{}^{(n+1)}C_2(2)\frac{d^nv}{dx^n}=2n\left[x\frac{d^{n+1}v}{dx^{n+1}}+{}^{(n+1)}C_1(1)\frac{d^nv}{dx^n}\right]$$

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}}+2x[{}^{n+1}C_1-n]\frac{d^{n+1}v}{dx^{n+1}}+2[{}^{n+1}C_2-n.{}^{(n+1)}C_1]\frac{d^nv}{dx^n} = 0$$

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}}+2x\frac{d^{n+1}v}{dx^{n+1}}-n(n+1)\frac{d^nv}{dx^n} = 0$$

$$(x^2-1)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-n(n+1)y = 0$$

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0$$

This shows that $y = \frac{d^n v}{dx^n}$ is a solution of Legendre's equation.

$$C\frac{d^n v}{dx^n} = P_n(x)$$

$$v = (x^2-1)^n = (x+1)^n (x-1)^n$$

$$\frac{d^n v}{dx^n} = (x+1)^n \frac{d^n}{dx^n}(x-1)^n + {}^nC_1 \cdot n(x+1)^{n-1} \cdot \frac{d^{n-1}}{dx^{n-1}}(x-1)^n + \ldots + (x-1)^n \frac{d^n}{dx^n}(x+1)^n = 0$$

$$\frac{d^n v}{dx^n} = 2^n \cdot n!$$

$$C \cdot 2^n \cdot n! = P_n(1) = 1 \qquad P_n(1) = 1$$

$$C = \frac{1}{2^n \cdot n!}.$$

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n v}{dx^n}$$

$$P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$$

Example 1. We define P(x) to be the degree n Legendre polynomial. Establish the continuity of the nth derivative for any function, f(x), by showing

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \frac{(-1)^n}{2^n\, n!}\int_{-1}^{1} (x^2-1)^n f^n(x)\, dx.$$

Solution

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \int_{-1}^{+1} f(x) \cdot \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n\, dx$$

$$\left[P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n\right]$$

$$= \frac{1}{2^n \lfloor n}\int_{-1}^{+1} f(x) \cdot \frac{d^n}{dx^n}(x^2-1)^n\, dx$$

$$= \frac{1}{2^n \lfloor n}\left[f(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \int f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx\right]_{-1}^{+1}$$

$$= \frac{1}{2^n \lfloor n}\left[0 - \int_{-1}^{+1} f'(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx\right]$$

$$= \frac{(-1)}{2^n \lfloor n}\int_{-1}^{+1} f'(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx$$

$$= \frac{(-1)}{2^n \lfloor n}\left[f'(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n - \int f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx\right]_{-1}^{+1}$$

$$= \frac{(-1)^2}{2^n \lfloor n}\int_{-1}^{+1} f''(x)\frac{d^{n-2}}{dx^{n-2}}\cdot(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n \lfloor n}\int_{-1}^{+1} f^n(x)\,(x^2-1)^n\,dx$$

Example 2: Express f(x) = $4x^2+6x^2+7x+2$ in terms of Legendre Polynomials.

Solution

$$4x^3+6x^2+7x+2 \equiv a\,P_3(x)+b\,P_2(x)+c\,P_1(x)+d\,P_0(x)$$

$$\equiv a\left(\frac{5x^3}{2}-\frac{3x}{2}\right)+b\left(\frac{3x^2}{2}-\frac{1}{2}\right)+c\,(x)+d\,(1)$$

$$\equiv \frac{5ax^3}{2}-\frac{3ax}{2}+\frac{3bx^2}{2}-\frac{b}{2}+cx+d$$

$$\equiv \frac{5ax^3}{2}+\frac{3bx^2}{2}+\left(\frac{-3a}{2}+c\right)x-\frac{b}{2}+d.$$

$$4 = \frac{5a}{2}, \quad \text{or} \quad a = \frac{8}{5}$$

$$6 = \frac{3b}{2} \quad \text{or} \quad b = 4$$

$$7 = \frac{-3a}{2}+c \quad \text{or} \quad 7 = \frac{-3}{2}\left(\frac{8}{5}\right)+c \quad \text{or} \quad c = \frac{47}{5}$$

$$2 = \frac{-b}{2}+d \quad \text{or} \quad 2 = \frac{-4}{2}+d \quad \text{or} \quad d = 4$$

$$4x^3+6x^2+7x+2 = \frac{8}{5}P_3(x)+4\,P_2(x)+\frac{47}{5}P_1(x)+4\,P_0(x)$$

6.3 Summary

Legendre functions, which include Legendre polynomials and associated Legendre functions, are crucial in mathematical physics and engineering due to their orthogonal properties and solutions to specific differential equations. Here’s a concise summary:

- Solutions to Legendre's differential equation

$$(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0$$

for non-negative integer n.

- Solutions to the associated Legendre differential equation, extending Legendre polynomials to functions involving an additional integer parameter m.
- Associated Legendre functions have orthogonality properties in the context of spherical harmonics.
- The generating function for Legendre polynomials is:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

- Legendre polynomials satisfy recurrence relations such as:

6.4 Keywords

- Legendre polynomials
- Orthogonality
- Rodrigue's formula
- Generating Function

6.5 Self Assessment

1. What is the differential equation satisfied by Legendre polynomials $P_n(x)$?
2. Over which interval are Legendre polynomials $P_n(x)$ orthogonal?
3. What is Rodrigue's formula for Legendre polynomials?
4. What is the generating function for Legendre polynomials?
5. Provide one of the recurrence relations for Legendre polynomials.
6. How are associated Legendre functions $P_n^m(x)$ related to Legendre polynomials?
7. What weight function is used in the orthogonality condition for Legendre polynomials?
8. In what fields are Legendre polynomials commonly applied?

6.6 Case Study

Explain how Legendre polynomials are used to solve Laplace's equation in spherical coordinates. Provide an example where the potential outside a spherical charge distribution is expressed in terms of Legendre polynomials.

6.7 References

- Brunt, B. v. (2014). The Calculus of Variations. Germany: Springer New York.
- Rukmangadachari, E. (2021). Engineering Mathematics - II:. India: Pearson Education India

Unit – 7
Gamma and Beta Functions

Learning Objective

- Students should understand the definitions of the Gamma function (Γ(x)) and Beta function (B(x, y)), their domains, and key properties such as recursion relations, integral representations, and special values (e.g., Γ(1/2), B(1, 1)).
- Mastery of the relationships between Gamma and Beta functions, such as:
- Understanding how Gamma and Beta functions are applied in various branches of mathematics, including:

Structure

7.1 Gamma Function (Γ(x))

Definition: The gamma function is defined for all complex numbers x except non-positive integers ($x \leq 0, x \in Z^-$). It is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

x> 0

Properties: There are two important properties:

$$\Gamma(n) = (n-1)!.$$

$$\Gamma(x+1) = x\Gamma(x).$$

Applications:

- Probability theory: Gamma distributions arise naturally as models for the sum of exponentially distributed random variables.
- Statistical mechanics: Appears in partition functions and calculating probabilities of states in quantum mechanics.
- Engineering: Used in solving differential equations and in Fourier transform pairs involving exponential and power functions.

7.2 Beta Function (B(x, y))

Definition: The beta function is defined as:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

where x>0 and y>0 .

Properties: There are two important properties:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$B(x,y) = B(y,x)$$

Applications:

- Statistical distributions: The Beta distribution is widely used in Bayesian statistics, especially as a prior distribution for probabilities.
- Physics: Appears in various areas of physics, including plasma physics, where it describes the distribution of particles.
- Engineering: Used in areas such as reliability analysis and control systems, particularly in defining probability density functions.

Relationship between Gamma and Beta Functions:

- Both functions are interconnected through the relation which underscores their complementary roles in mathematical analysis.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

- They are crucial in integrating functions involving exponential and trigonometric functions, making them versatile tools in mathematical analysis.

Example 1. Prove that

$\Gamma 1 = 1$

Solution:

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$$\Gamma 1 = \int_0^{\infty} e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

Example 2. Prove that

(*i*) $\Gamma(n+1) = n\,\Gamma n$ (*ii*) $\Gamma(n+1) = \lfloor n$

Solution:

(*i*) $$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x}\, dx \qquad ...(1)$$

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^{\infty} - (n-1)\int_0^{\infty} x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \dots + \frac{x^n}{\lfloor n} + \dots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^{\infty} x^{n-2} e^{-x}\, dx$$

$$\therefore \qquad \Gamma n = (n-1)\,\Gamma(n-1) \qquad ...(2)$$

$$\Gamma(n+1) = n\,\Gamma n$$ Replacing n by $(n+1)$ **Proved**

(*ii*) Replace n by $n-1$ in (2), we get

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

$$\Gamma n = (n-1)(n-2)\,\Gamma(n-2)$$

$$\Gamma n = (n-1)(n-2)\dots 3.2.1\,\Gamma 1$$

$$\Gamma n = (n-1)(n-2)\ldots 3.2.1.1$$

$$\Gamma n = \underline{|n-1}$$

n by $n + 1$, we have

$$\overline{|n+1} = \underline{|n}$$

Example:3 Find the value of

$$\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$$

Solution:

$$I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$$

$$\sqrt[3]{x} = t \text{ or } x = t^3 \text{ or } dx = 3\,t^2\,dt$$

$$I = \int_0^\infty t^{3/2} e^{-t}\, 3\, t^2\, dt = 3\int_0^\infty t^{7/2} e^{-t}\, dt = 3\,\Gamma\frac{9}{2} = 3\,.\,\frac{7}{2}\,.\,\frac{5}{2}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi}$$

Example: 4 Prove that

$$\beta(l, m) = \frac{\Gamma l\, \Gamma m}{|l+m}$$

Proof: $\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\, dx = \int_0^1 (1-x)^{m-1} x^{l-1}\, dx$

$$= \left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\left(\frac{x^l}{l}\right) dx$$

$$= \frac{(m-1)}{l}\int_0^1 (1-x)^{m-2} x^l\, dx$$

$$= \frac{(m-1)(m-2)}{l(l+1)}\int_0^1 (1-x)^{m-3} x^{l+1}\, dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\int_0^1 x^{l+m-2}\, dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\left[\frac{x^{l+m-1}}{l+m-1}\right]_0^1$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots(l+m-2)(l+m-1)}$$

$$= \frac{\underline{|m-1}}{l\,(l+1)\ldots(l+m-2)\,(l+m-1)} \times \frac{(l-1)\,(l-2)\ldots 1}{(l-1)\,(l-2\ldots 1}$$

$$= \frac{\underline{|m-1}\ \underline{|l-1}}{1.2\ldots(l-2)\,(l-1)\,.\,l\,(l+1)\ldots(l+m-2)\,(l+m-1)}$$

$$= \frac{\underline{|l-1}\ \underline{|m-1}}{\underline{|l+m-1}}$$

$$= \frac{\Gamma l\, \Gamma m}{\Gamma l+m}$$

Example 5: Find the value of

$$\int_0^1 \frac{x^{m-l} + x^{n-l}}{(1+x)^{m+n}}\, dx$$

Solution:

$$\beta\,(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \beta\,(m, n)$$

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta\,(m, n)$$

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad\qquad \left(\text{Put } x = \frac{1}{t}\right)$$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\, dt\right) = \int_0^1 \frac{\left(\frac{1}{t}\right)^{m-1} \frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n} (t+1)^{m+n}}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \text{ in (1) we get}$$

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \beta\ (m, n)$$

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = \beta\ (m, n)$$

Example 6 : Find the value of $\sqrt{\frac{1}{2}}$

Solution:

$$\int_0^{\frac{\pi}{2}} \sin^P\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)}$$

$$\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(1)}$$

$$[\,\theta\,]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \quad \text{or} \quad \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \quad \text{or} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 7: Show that

$$\int_0^{\frac{\pi}{2}} \sqrt{(\cot\theta)}\, d\theta = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Solution:

$$\int_0^{\frac{\pi}{2}} \sin^P x \cos^q x \, dx = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{1/2+1}{2}\right)}{2\,\Gamma\left(\frac{-1/2+1/2+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)} = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

7.3 Summary

- The Gamma function, denoted as $\Gamma(x)$, extends the factorial function to real and complex numbers (except non-positive integers).
- Probability theory: Defines the Gamma distribution, used in modeling waiting times and sum of exponential random variables.
- Statistical mechanics: Appears in partition functions and statistical distributions.
- Engineering: Utilized in solving differential equations, particularly those arising in heat conduction and wave propagation.
- Statistical distributions: Beta distribution is fundamental in Bayesian statistics and modeling proportions.

7.4 Keywords

- Beta function
- Gamma function
- Special Functions

7.5 Self Assessment

1. Define the Gamma function ($\Gamma(x)$) and state its key property.
2. What is the integral representation of the Beta function ($B(x, y)$)?
3. Explain the relation between Gamma and Beta functions.
4. State one important application of the Gamma function in probability theory.
5. What symmetry property does the Beta function possess?
6. How are Gamma and Beta functions used in engineering applications?
7. What is the special value of $\Gamma(1/2)$?
8. Describe a scenario where Beta functions are commonly applied in statistics.

7.6 Case Study

A financial analyst is modeling the distribution of stock returns over a given period using probability distributions.

- Explain how the Gamma function and its relation to the Gamma distribution are used to model waiting times between events in financial markets.

- How could the Beta function be applied in Bayesian inference to model prior distributions of asset returns?

7.7 References

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Unit - 8
Theory of Errors

Learning objectives

- Students learn about the inherent uncertainties and errors associated with measurements in scientific experiments and real-world applications. This includes differentiating between systematic errors (bias) and random errors (precision).
- Mastery of how errors propagate through mathematical operations (such as addition, subtraction, multiplication, division) and composite functions is essential. This involves understanding methods like propagation of uncertainties using derivatives or statistical approaches.

Structure

8.1 Introduction

When students consistently make mistakes, one technique that is frequently utilized to determine the reason behind their blunders is error analysis. It involves going over a student's work and searching for any areas where there can be misunderstandings. Mathematical errors can be factual, procedural, or conceptual, and they can happen for several reasons.

Important of error analysis:

For a number of reasons, error analysis is essential in mathematics and other scientific fields. They are as follow:

1. **Accuracy and Reliability**: The accuracy and dependability of theoretical forecasts and numerical findings are guaranteed by error analysis. Researchers are able to evaluate the reliability of their results and conclusions by identifying possible sources of inaccuracy and measuring the impact of those sources.

2. **Decision Making**: Precise computations are necessary for well-informed decision-making in practical applications including engineering, finance, and medicine. Error analysis supports decision-making, risk assessment, and process optimization for stakeholders.

3. **Validation of Models**: Through the comparison of anticipated consequences with observed data or experimental results, error analysis verifies mathematical models. Differences between predictions and observations frequently point to areas that require modification in the experimental setup or where models need to be improved.

4. **Uncertainty Quantification**: It measures the degree of uncertainty in calculations and measurements. Researchers can explain the trustworthiness of their data and make well-informed decisions based on uncertainty levels when they have a clear understanding of the range and size of mistakes.

5. **Optimization and Improvement**: Researchers can concentrate on streamlining processes, honing algorithms, or enhancing experimental methods in order to reduce errors and increase accuracy by determining the primary causes of errors.

6. **Ethical Considerations**: Transparency and moral behavior in research are guaranteed by careful error analysis. It supports the integrity of scientific research by preventing false findings or misinterpretations that can result from uncontrolled errors.

Error analysis is crucial because it supports the validity and practicality of mathematical applications and scientific research, to put it briefly. It guarantees that inferences made from data or theoretical models are solid, reliable, and accurate representations of the phenomena being studied.

Three categories are used to classify the errors in mathematics:

i) **Factual errors:** These errors arise when students fail to grasp a fact that is necessary to answer the problem.

ii) **Procedural errors:**These errors arise when pupils apply mathematical operations incorrectly.

iii) **Conceptual errors:** These errors stemming from erroneous beliefs or understandings of the ideas and concepts associated with the issue.

8.2 Significance of Digits

In many scientific and mathematical contexts, the importance of digits—often discussed in relation to important figures or significant digits—is paramount. Here are the main explanations for the significance of numbers:

1. **Precision and Accuracy**: The accuracy of a calculation or measurement is indicated by its digits. The degree of confidence in the measurement or the precision of the computed result is indicated by the number of significant digits. More significant digits, for example, indicate a more accurate measurement or computation.
2. **Error Estimation**: Significant numbers are useful for calculating the extent of measurement or computation errors or uncertainties.
3. **Communication of Results**: Using the right amount of significant digits in scientific and technical communication guarantees clarity and prevents erroneous interpretations. It communicates the degree of confidence and dependability of the reported facts or outcomes.
4. **Propagation of Uncertainty**: The principles of significant figures govern how uncertainties spread through mathematical operations when doing calculations involving measurements with errors.
5. **Standardization:** In many scientific domains, significant figures offer a common language for expressing numerical quantities.
6. **Experimental Design**: In experimental design, understanding significant digits helps researchers determine the necessary precision of instruments and the required number of measurements to achieve reliable results within a specified level of uncertainty.

7. **Quality Control**: Significant numbers are important for quality control in industries like manufacturing and engineering.
8. **Mathematical and Computational Accuracy**: The proper use of significant digits in computational mathematics guarantees that numerical algorithms yield accurate answers without adding unnecessary computational mistakes from truncation or rounding.

In general, the importance of numbers comes from their ability to convey accuracy, control uncertainty, enable precise communication, and guarantee the accuracy of technical and scientific data and computations.

Example 1:

A student measures the length of a table using a ruler that measures to the nearest millimeter. The measurement obtained is 2.395 meters. Determine the number of significant digits in this measurement and explain their significance.

Solution:

The measurement given is 2.395 meters. To determine the number of significant digits:

Identify Significant Digits:Significant digits are the digits that contribute to the precision of the measurement.

In the number 2.395, there are four digits: 2, 3, 9, and 5.

The zero at the end (after the decimal point) is also considered significant because it indicates the precision of the measurement.

Count Significant Digits:

Counting the digits in 2.395 gives us four significant digits: 2, 3, 9, and 5.

Significance of Significant Digits:

The significant digits (2.395) indicate that the measurement was recorded with a precision to the nearest millimeter (since 1 meter = 1000 millimeters).

Each significant digit reflects the accuracy of the measurement. Here, the measurement is precise to the nearest millimeter, which is important for applications where precise dimensions are needed.

Therefore, the measurement of 2.395 meters has four significant digits: 2, 3, 9, and 5. These digits convey the precision of the measurement and are essential for accurately representing the length of the table.

This example illustrates how significant digits are determined and why they are important in accurately representing measurements with a specified level of precision. It also emphasizes the role of significant digits in conveying the reliability and accuracy of numerical data.

Precision of a number:

The precision of a number refers to the degree of exactness or detail to which it is expressed. In mathematics and science, precision is crucial for accurately representing measurements, calculations, and theoretical values. Here are key points regarding the precision of numbers:

1. **Number of Decimal Places: A** number's precision is frequently expressed as the number of decimal places it contains. For instance, because 3.14159 has more digits after the decimal point than 3.14, it is more precise than 3.14.
2. **Significant Figures:** Another technique to convey precision is through significant figures, often known as important digits. They consist of one ambiguous digit in addition to all the recognized digits. For example, 123.45 is made up of five significant digits.
3. **Scientific Notation:** When expressing numbers in a condensed form and emphasizing their precision, scientific notation is frequently utilized. For instance, $1.23 \times 10^{(-4)}$ can be written as the number 0.000123, signifying three significant figures.
4. **Contextual Precision:** Depending on the situation, a number's necessary precision varies. Certain computations or comparisons only require a high degree of precision, while in other situations a rough estimate with a few key values is sufficient.
5. **Measurement Tools:** The sensitivity and accuracy of the tools employed determine how precise the measurements are frequently. Higher precision instruments may measure in smaller increments, resulting in numerical values that are more precise.
6. **Computational Precision:** The quantity of significant digits utilized in computations is referred to as computational precision in mathematics. It has an impact on the precision of numerical techniques and algorithms, particularly when working with extremely big or extremely small quantities or in iterative processes.

For efficient communication in scientific research, engineering, finance, and many other disciplines where accuracy and dependability are critical, one must comprehend and accurately explain the precision of numbers.

Example 2 : A scientist measures the diameter of a sphere to be 10.5 centimeters. The diameter is then used to calculate the volume of the sphere using the formula$V = \frac{4}{3}\pi r^3$, where r is half of the diameter. Determine the volume of the sphere, considering the precision of the diameter measurement.

Solution.

Identify the Precision of the Diameter:

The diameter measurement given is 10.5 centimeters.

Calculate the Radius:

The radius r of the sphere is half of the diameter: $r = \frac{10.5}{2}$ cm = 5.25 cm

Note that the radius calculation retains the precision of one decimal place, consistent with the diameter measurement.

Compute the Volume of the Sphere:

Use the formula for the volume of a sphere: $V = \frac{4}{3}\pi r^3$

Substitute the radius r=5.25 cm into the formula: $V = \frac{4}{3}. 3.14. (2.25)^3$

Calculate $(5.25)^3$ i. e. 144.703125

Now, calculate the volume V:

$V \approx \frac{4}{3}\pi \times 144.703125 \approx 957.6$ cubic centimeters'

Consider the Precision of the Volume:

The calculated volume 957.6cubic centimeters should be reported to match the precision of the given diameter measurement.

Since the diameter was measured to one decimal place (indicating a precision of ±0.05cm), the volume should ideally be reported to one decimal place as well: 957.6cubic centimeters.

Therefore, the volume of the sphere, considering the precision of the diameter measurement, is 957.6 cubic centimeters. This ensures that the calculated result aligns with the precision of the initial measurement and maintains accuracy in the final answer.

This example demonstrates how to calculate a quantity (volume in this case) while considering the precision of the initial measurement (diameter). It emphasizes the importance of maintaining consistent precision throughout calculations to ensure accurate and reliable results.

8.3 Types of Error

In a general manner, errors are basically of two types:

- Systematic Errors
- Random Errors

Systematic Errors: Errors that exclusively happen in one direction are referred to as systematic errors. Either a positive or negative direction is possible, but not simultaneously. Since repetitive errors are caused by default machinery and improper experiment apparatus, systematic errors are sometimes referred to as repetitive errors. These mistakes occur when the measurement instrument is not properly calibrated. The following are a few sources of systematic errors:

Random Errors

Random Errors rely on measurements to measurements and are not fixed on general perimeters. Because they are random in nature, that is why they are called random errors. Variations in statistical readings brought on by the instrument's precision limitations are another name for random mistakes. Random errors can arise from sudden and unforeseen changes in the environment of the experiment.

Example3:

A spring balance that is not in a constant temperature environment will produce variable results. An individual is more likely to obtain different results from an experiment if they repeat it.

Because random errors are unpredictable and not fixed in nature like systematic errors are, we can only lessen them; we cannot totally remove them.

Least Count

Least Count of the Instrument is the least value that may be measured in an instrument. The primary component of a measurement is defined by the least count, which can occur in both systematic and random errors.

The instrument's resolution affects the least count error. If we are aware of the observations and the least count of instruments, we can compute the Least Count Error. The least count of several instruments is displayed in the table below.

Instrument	Micrometer	Sphero meter	Vernier Caliper
Least Count	0. 0001cm	0.001cm	0.01 cm

We reduce least count error by using high-precision sensors to refine experiment designs. We repeat the experiment multiple times and take the arithmetic mean of all the observations to reduce least count error. The measurement's true value is never far from the mean value.

Combination of Errors

Performing a physics experiment involves dealing with various errors. The mistakes could be in multiplication or division, addition, or subtraction. Since pressure, for instance, is defined as force per unit area, there is a possibility that pressure will also be erroneous if there are errors in force and area. How can I now compute that error? The computation of combined errors can be done in two ways:

- Error of a sum or difference
- Error in product or quotient
- Error in case of a measured quantity raised to a power
- Error of a sum or difference

If two physical quantities, X and Y, have actual values of $X \pm \Delta X$ and $Y \pm \Delta Y$, then $Z = X + Y$ can be used to compute the error in their total. The maximum mistake in Z is therefore $\Delta Z = \Delta X + \Delta Y$, and the same method can be used to calculate the difference. Keep in mind that the absolute error in the final solution is always the total of the individual absolute mistakes when two quantities are added or subtracted.

8.4 Summary

- The theory addresses the inherent uncertainties and errors associated with measurement processes. It distinguishes between systematic errors (bias) and random errors (precision), emphasizing the need to quantify and minimize both types to ensure accurate results.
- It deals with how uncertainties propagate through calculations and operations. Methods such as propagation of uncertainties using derivatives or statistical approaches are used to estimate the combined uncertainty in derived quantities.
- Statistical tools are employed to analyze measurement data, including concepts like probability distributions (such as Gaussian or normal distribution), standard deviation, variance, confidence intervals, and hypothesis testing. These techniques provide a framework for understanding the reliability and significance of experimental results.

8.5 Keywords

- Systematic Errors
- Random Errors
- Probability
- Distributions

8.6 Self Assessment

1. Define measurement uncertainty.
2. What are systematic errors? Provide an example.
3. How does random error differ from systematic error?
4. Explain error propagation in measurement.
5. Why is calibration important in error analysis?
6. What statistical measure is commonly used to quantify spread of data due to random errors?
7. How does error analysis contribute to quality control in manufacturing?
8. Give an example of applying error analysis in experimental science.

8.7 Case Study

A laboratory is analyzing water samples for pollutant concentrations using spectrophotometry.

- How would you assess and quantify the uncertainties associated with the spectrophotometric measurements?
- Discuss the sources of systematic and random errors that could affect the accuracy of pollutant concentration measurements.
- Explain how error propagation principles could be applied to estimate the overall uncertainty in the reported pollutant concentrations.

8.8 References

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Unit - 9

Introduction to Partial Differential Equations (PDEs)

Learning Objective

- Define and identify partial differential equations (PDEs) and distinguish them from ordinary differential equations (ODEs).
- Understand and compute partial derivatives of multivariable functions.
- Classify PDEs based on order, linearity, and homogeneity.

Structure

9.1 Partial Differential Equations

Equations that have independent, dependent, and partial differential coefficients are known as partial differential equations.
The dependent variable will be represented by z, and the independent variables by x and y. Here is how the partial differential coefficients are represented:

$$\frac{\partial z}{\partial x} = p, \qquad \frac{\partial z}{\partial y} = q.$$

$$\frac{\partial^2 z}{\partial x^2} = r, \qquad \frac{\partial^2 z}{\partial x \partial y} = s, \qquad \frac{\partial^2 z}{\partial y^2} = t$$

Classification of PDEs

PDEs can be classified based on their linearity, order, and the number of independent variables.

Linearity:

- **Linear PDEs:** The unknown function and its derivatives appear to the power of one (e.g., $u_{xx}+u_{yy}=0$.

- **Nonlinear PDEs:** The equation involves nonlinear terms of the unknown function or its derivatives (e.g., $u_t+uu_x=0$).

Order:

- **First-order PDEs:** Involve only the first derivatives of the unknown function (e.g., $u_x+u_y=0$).
- **Second-order PDEs:** Involve second derivatives (e.g., $u_{xx}+u_{yy}=0$).

Common Types of PDEs

1. **Heat Equation:** $u_t=\alpha u_{xx}$

   Describes how heat is distributed, or how temperature changes over time, in a certain area.
2. **Wave Equation:** $u_{tt}=c^2u_{xx}$

   Describes the propagation of waves, such as sound or light waves, through a medium.
3. **Laplace's Equation:** $\Delta u=0$ or $u_{xx}+u_{yy}+u_{zz}=0$

   A special case of the Poisson equation where there is no source term. It appears in problems of electrostatics, gravitation, and fluid flow.
4. **Poisson's Equation:** $\Delta u=f$ or $u_{xx}+u_{yy}+u_{zz} = f(x,y,z)$

   Generalization of Laplace's equation with a source term f.

9.2 Method of forming PDEs

A PDEs is formed by two methods.

(i) By eliminating arbitrary constants.

(ii) By eliminating arbitrary functions.

(i) Method of elimination of arbitrary constants

Example 1. Form a PDE from

$$x^2 + y^2 + (z-c)^2 = a^2 .$$

Solution

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$x + (z-c)p = 0$$

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$y + (z-c)q = 0$$

$$(z-c) = -\frac{x}{p}$$

$$y - \frac{x}{p}q = 0$$

$$yp - xq = 0$$

(ii) Method of elimination of arbitrary functions

Example 2. Form the PDE from

$$z = f(x^2 - y^2)$$

Solution

$$z = f(x^2 - y^2)$$

Differentiating w.r.t x and y

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\ 2x$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)\ (-2y)$$

$$\frac{p}{q} = \frac{-x}{y} \qquad \text{or} \qquad py = -qx$$

$$yp + xq = 0$$

Example 3: Solve

$$\frac{\partial^3 z}{\partial x^2 \partial y} = cos(2x + 3y)$$

Solution

$$z = -\frac{1}{12}\sin(2x + 3y) + x\int \phi(y)\,dy + \int g(y)\,dy$$

$$z = -\frac{1}{12}\sin(2x + 3y) + x\,\phi_1(y) + \phi_2(y)$$

Example 4: Solve

$$y^2p - xyq = x(z - 2y).$$

Solution

$$y^2p - xyq = x(z - 2y)$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad ...(1)$$

$$\frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad x\,dx = -y\,dy$$

Intregrating

$$\frac{x^2}{2} = -\frac{y^2}{2} + \frac{C_1}{2}$$

$$x^2 + y^2 = C_1 \qquad ...(2)$$

$$-\frac{dy}{y} = \frac{dz}{z-2y}$$

$$-z\,dy + 2y\,dy = y\,dz \quad \text{or} \quad 2y\,dy = y\,dz + z\,dy$$

On integration, we get

$$y^2 = yz + C_2$$

$$y^2 - yz = C_2 \qquad ...(3)$$

From (2) and (3)

$$x^2 + y^2 = f(y^2 - yz)$$

9.3 Summary

- **Partial Derivatives:** These are derivatives of functions with respect to one variable while keeping others constant.
- **PDEs:** Equations involving partial derivatives of an unknown function with respect to multiple independent variables.
- **Linearity:**
    - **Linear PDEs:** The unknown function and its derivatives appear to the first power.
    - **Nonlinear PDEs:** Involve nonlinear terms of the unknown function or its derivatives.
- **Order:**
    - **First-order PDEs:** Involve first derivatives.
    - **Second-order PDEs:** Involve second derivatives.

9.4 Keywords

- Partial Differential Equations
- Partial Derivatives
- First-order PDEs
- Second-order PDEs

9.5 Self Assessment

Solve the following

1. $p \tan x + q \tan y = \tan z$ **Ans.** $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$
2. $y \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z^2 + 1$ **Ans.** $f(x-y) = \log y - \tan^{-1} z$
3. $(y - z)\, p + (x - y)\, q = z - x$ **Ans.** $f(x + y + z\,,\ x^2 + 2yz)$
4. $(y + zx)\, p - (x + yz)\, q = x^2 - y^2$ **Ans.** $f(x^2 + y^2 - z^2) = (x - y)^2 - (z + 1)^2$
5. $zx \frac{\partial z}{\partial x} - zy \frac{\partial z}{\partial y} = y^2 - x^2$ **Ans.** $f(x^2 + y^2 + z^2\,, xy) = 0$
6. $pz - qz = z^2 + (x + y)^2$ **Ans.** $[\, z^2 + (x + y)^2\,]\, e^{-2x} = f(x + y)$
7. $p + q + 2xz = 0$ **Ans.** $f(x - y) = x^2 + \log z$

9.6 Case Study

Consider a long, thin rod of length L with insulated sides. The temperature distribution in the rod over time is governed by the heat equation $u_t = \alpha u_{xx}$, where α\alphaα is the thermal diffusivity constant.

- Assume that u(x,0)=f(x) represents the initial temperature distribution along the rod. u(0,t)=0 and u(L,t)=0, or the ends of the rod, are maintained at a constant temperature of 0 degrees. For this case, formulate the starting and boundary value problems.
- For the above boundary conditions, determine the general form of the solution u(x,t) using the separation of variables approach.

9.7 References

- Strauss, W. A. (2008). Partial Differential Equations: An Introduction. United Kingdom: Wiley.
- Hillen, T., Leonard, I. E., van Roessel, H. (2014). Partial Differential Equations: Theory and Completely Solved Problems. Germany: Wiley.

Unit – 10

Solutions of PDEs

Learning Objective

- Solve PDEs using separation of variables, and understand when this method is applicable.
- Apply Fourier series and Fourier transform methods to solve PDEs.
- Utilize Green's functions to solve linear non-homogeneous PDEs.
- Understand the method of characteristics for solving first-order PDEs.
- Define and identify partial differential equations (PDEs) and distinguish them from ordinary differential equations (ODEs).
- Understand and compute partial derivatives of multivariable functions.
- Classify PDEs based on order, linearity, and homogeneity.

Structure

10.1 Introduction

In practical problems, the following types of equations are generally used:

1. **Wave equation:**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2. **One-dimensional heat flow:**

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

3. **Two-dimensional heat flow:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4. **Radio equations:**

$$-\frac{\partial V}{\partial x} = L\frac{\partial I}{\partial t}, \quad -\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t}$$

10.2 Method of Separation of Variables

In this method, two functions are used to determine the dependent variable, and each function only considers one of the independent variables. This leads to the creation of two common type differential equations.

Example 1. Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

$$u(x, 0) = 6e^{-3x}$$

Solution

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad ...(1)$$

$$u = X(x).T(t) \qquad ...(2)$$

Putting the value of u in (1) we get

$$\frac{\partial (X.T)}{\partial x} = 2\frac{\partial}{\partial t}(X.T) + X.T$$

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + X.T \quad \text{or} \quad T.X' = 2X.T' + X.T \quad \text{or} \quad T\frac{X'}{X} = 2\frac{T'}{T} + 1 = c$$

(a) $\frac{X'}{X} = c$ or $\frac{1}{X}\frac{dX}{dx} = c$ or $\frac{dX}{X} = c\,dx$

On integration $\log X = cx + \log a$. or $\log \frac{X}{a} = cx$ or $\frac{X}{a} = e^{cx}$ or $X = ae^{cx}$

(b) $\frac{2T'}{T} + 1 = c$ or $\frac{T'}{T} = \frac{1}{2}(c-1)$ or $\frac{1}{T}\frac{dT}{dt} = \frac{1}{2}(c-1)$ or $\frac{dT}{T} = \frac{1}{2}(c-1)\,dt$

On integration $\log T = \frac{1}{2}(c-1)t + \log b$ or $\log \frac{T}{b} = \frac{1}{2}(c-1)t$

or $\frac{T}{b} = e^{\frac{1}{2}(c-1)t}$ or $T = be^{\frac{1}{2}(c-1)t}$

$$u = ae^{cx} . be^{\frac{1}{2}(c-1)t}$$

or $u = ab\ e^{cx + \frac{1}{2}(c-1)t}$ ...(3)

or $u(x, 0) = ab\ e^{cx}$

But $u(x, 0) = 6\,e^{-3x}$

i.e. $ab\,e^{cx} = 6\,e^{-3x}$ or $ab = 6$ and $c = -3$

Putting the value of ab and c in (3), we have

$$u = 6\,e^{-3x + \frac{1}{2}(-3-1)t}$$

$$u = 6\,e^{-3x-2t}$$

10.3 Solution of Wave Equations

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{... (1)}$$

Let $u = x + ct$, $v = x - ct$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u}(1) + \frac{\partial y}{\partial v}(1) = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial^2 y}{\partial v^2} \qquad \text{... (2)}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial y}{\partial u}c + \frac{\partial y}{\partial v}(-c) = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \left[\because \frac{\partial u}{\partial t} = c,\ \frac{\partial v}{\partial t} = -c\right]$$

$$\frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial^2 y}{\partial v^2}\right) \qquad \text{... (3)}$$

Substituting the values, of $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ from (2) and (3) in (1), we get

$$\left(\frac{d^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\, \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\, \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \quad \text{or} \quad \frac{\partial^2 y}{\partial u\, \partial v} = 0 \qquad \text{... (4)}$$

Integrating (4) ... (5)

$$y = \int f(u)\, du + \psi(v)$$

$$y = \phi(u) + \psi(v) \quad \text{where } \phi(u) = \int f(u)\, du$$

$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \text{...(6)}$$

Differentiating (6) *w.r.t.* "*t*" , we get

$$\frac{\partial y}{\partial t} = c\, \phi'(x + ct) - c\, \psi'(x - ct) \qquad \text{... (7)}$$

$$0 = c\, \phi'(x) - c\psi'(x)$$

$$\phi'(x) = \psi'(x) \quad \text{or} \quad \phi(x) = \psi(x) + b$$

Again substituting $y = f(x)$ and $t = 0$ in (6) we get

$$f(x) = \phi(x) + \psi(x) \quad \text{or} \quad f(x) = [\psi(x) + b] + \psi(x)$$

$$f(x) = 2\psi(x) + b$$

On putting the values of $\phi(x + ct)$ and $\psi(x - ct)$ in (6),

$$y(x,t) = \frac{1}{2}[f(x + ct) + b] + \frac{1}{2}[f(x - ct) - b]$$

$$y(x,t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

10.4 Equation Of Vibrating String

Think of a taut elastic string that is stretched between points O and A. Assign O to the origin and OA to the x-axis. when the string is slightly moved perpendicular to its length (i.e., parallel to the y-axis). At any given moment, let y represent the displacement at point P (x, y). The equation for waves.

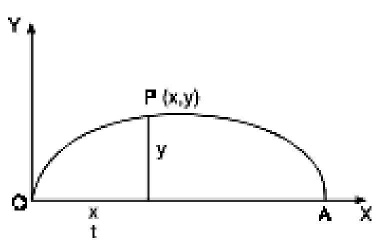

Figure 10.1 : Trajectory of Vibrating String

Example 2. Obtain the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Solution

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial y}{\partial t} = X \frac{dT}{dt} \quad \text{and} \quad \frac{\partial y}{\partial x} = T \frac{dX}{dx}$$

$$\frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

$$X \frac{d^2 T}{d t^2} = c^2 T \frac{d^2 X}{d x^2}$$

By separating the variables, we get

$$\frac{\frac{d^2 T}{dt^2}}{c^2 T} = \frac{\frac{d^2 X}{dx^2}}{X} = k$$

$$\frac{d^2 T}{d t^2} - k\, c^2 T = 0 \quad \text{and} \quad \frac{d^2 X}{d x^2} - k\, X = 0$$

$$m^2 - k\, c^2 = 0 \quad \text{or} \quad m = \pm\, c \sqrt{k} \quad \text{and} \quad m^2 - k = 0 \quad \text{or} \quad m = \pm \sqrt{k}$$

Case 1: If k >0

$$T = C_1 e^{c\sqrt{k}\, t} + C_2 e^{-c\sqrt{k}\, t}$$
$$X = C_3 e^{\sqrt{k}\, x} + C_4 e^{-\sqrt{k}\, x}$$

Case 2: If k <0

$$T = C_5 \cos c \sqrt{k}\, t + C_6 \sin c \sqrt{k}\, t$$
$$X = C_7 \cos \sqrt{k}\, x + C_8 \sin \sqrt{k}\, x$$

Case 3: If k =0

$$T = C_9\, t + C_{10}$$
$$X = C_{11} x + C_{12}$$

$$y = TX$$

$$y = (C_5 \cos c\sqrt{k}\,t + C_6 \sin c\sqrt{k}\,t) \times (C_7 \cos \sqrt{k}\,x + c_8 \sin \sqrt{k}\,x)$$

10.5 Laplace Equation In Polar Co-Ordinates

Example 3. Solve by the method of separation of variables.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solution

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{or} \quad r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$$

Let $u = R(r).\,T(\theta)$

$$\frac{\partial u}{\partial r} = \frac{dR}{dr}.\,T(\theta) \quad \text{and} \quad \frac{\partial^2 u}{\partial r^2} = \frac{d^2R}{dr^2}.\,T(\theta)$$

$$\frac{\partial u}{\partial \theta} = R(r).\,\frac{dT}{d\theta} \quad \text{and} \quad \frac{\partial^2 u}{\partial \theta^2} = R(r).\,\frac{d^2T}{d\theta^2}$$

$$r^2.\frac{d^2R}{dr^2}.\,T(\theta) + r\frac{dR}{dr}.\,T(\theta) + R(r)\frac{d^2T}{d\theta^2} = 0$$

$$\left(r^2\frac{d^2R}{dr^2} + r.\frac{dR}{dr}\right).T + R\frac{d^2T}{d\theta^2} = 0$$

$$\frac{r^2\frac{d^2R}{dr^2} + r.\frac{dR}{dr}}{R} = -\frac{1}{T}\frac{d^2T}{d\theta^2} = h$$

$r^2\frac{d^2R}{dr^2} + r.\frac{dR}{dr} - hR = 0$	$\frac{d^2T}{d\theta^2} + hT = 0$
Put $r = e^z$	$(D^2 + h)\,T = 0$
$(D(D-1) + D - h)\,R = 0$	$D^2 + h = 0$ or $D = \pm\, i\sqrt{h}$
$D^2 - h = 0 \rightarrow D = \pm\sqrt{h}$	$T = c_3 \cos(\sqrt{h}\,\theta) + c_4 \sin(\sqrt{h}\,\theta)$
$R = c_1 e^{\sqrt{h}z} + c_2 e^{-\sqrt{h}z}$	
$R = c_1 r^{\sqrt{h}} + c_2 r^{-\sqrt{h}}$	

$$u = (c_1 r^{\sqrt{h}} + c_2 r^{-\sqrt{h}})[c_3 \cos(\sqrt{h}\,\theta) + c_4 \sin(\sqrt{h}\,\theta)] \qquad ...(2)$$

$$u = (c_1 r^k + c_2 r^{-k})\,[c_3 \cos(k\theta) + c_4 \sin(k\theta)]$$

$$u = (c_5 + c_6 \log r)\,(c_7 + c_8\,\theta)$$

$$u = [c_9 \cos(p \log r) + c_{10} \sin(p \log r)]\,[c_{11} e^{p\theta} + c_{12} e^{-p\theta}]$$

10.6 Summary

The solutions of PDEs are crucial for modeling and understanding a wide range of physical phenomena. Analytical methods provide exact solutions and deeper insights into the structure of PDEs, while numerical methods offer powerful tools for solving complex problems that are analytically intractable. Understanding the strengths and limitations of each method is key to effectively applying PDEs in science and engineering.

10.7 Keywords

- Laplace Equation
- Wave Equations
- Separation of Variables
- Vibrating String

10.8 Self Assessment

Solve the following

1. $2x \dfrac{\partial z}{\partial x} - 3y \dfrac{\partial z}{\partial y} = 0$ **Ans.** $z = c\, x^{\frac{k}{2}}\, y^{\frac{k}{3}}$
2. $\dfrac{\partial u}{\partial x} + u = \dfrac{\partial u}{\partial t}$ if $u = 4e^{-3x}$ when $t = 0$ **Ans.** $u = 4e^{-3x-2t}$
3. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$ and $u = e^{-5y}$ when $x = 0.$ **Ans.** $u = e^{2x-5y}$
4. $4\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 3u,\ u = 3\,e^{-x} - e^{-5x}$ at $t = 0$ **Ans.** $u = 3\,e^{t-x} - e^{2t-5x}$
5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0;\quad u(x,0) = 4\,e^{-x}$ **Ans.** $u = 4\,e^{-x+3/2\,y}$
6. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2\,(x+y)\,u$ **Ans.** $u = ce^{x^2+y^2+k(x-y)}$

10.9 Case Study

Consider a rectangular region $0 \le x \le a_0$ and $0 \le y \le b_0$ with the electrostatic potential u(x,y) satisfying Laplace's equation $u_{xx}+u_{yy}=0$.

Suppose the potential is specified on all four boundaries: $u(0,y)=f_1(y)$, $u(a,y)=f_2(y)$, $u(x,0)=g_1(x)$, and $u(x,b)=g_2(x)$. Formulate the boundary value problem for u(x,y).

By Using method of separation of variables to solve the problem. Assume u(x,y)=X(x)Y(y) and derive the resulting ODEs for X(x) and Y(y).

10.10 References

- Strauss, W. A. (2008). Partial Differential Equations: An Introduction. United Kingdom: Wiley.
- Hillen, T., Leonard, I. E., van Ressel, H. (2014). Partial Differential Equations. Germany: Wiley.